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The E_{11} origin of all maximal supergravities. The hierarchy of field-strengths

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Abstract: Starting from E_{11} and the space-time translations we construct an algebra that promotes the global E_{11} symmetries to local ones, and consider all its possible massive deformations. The Jacobi identities imply that such deformations are uniquely determined by a single tensor that belongs to the same representation of the internal symmetry group as the $D-1$ forms specified by E_{11}. The non-linear realisation of the deformed algebra gives the field strengths of the theory which are those of any possible gauged maximal supergravity theory in any dimension. All the possible deformed algebras are in one to one correspondence with all the possible massive maximal supergravity theories. The hierarchy of fields inherent in the E_{11} formulation plays an important role in the derivation. The tensor that determines the deformation can be identified with the embedding tensor used previously to parameterise gauged supergravities. Thus we provide a very efficient, simple and unified derivation of all the field strengths and gauge transformations of all maximal gauged supergravities from E_{11}. The dynamics arises as a set of first order duality relations among these field strengths.

Keywords: Extended Supersymmetry, Gauge Symmetry, Supergravity Models

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Contents

1 Introduction

The maximal supergravity theories have played a key role in our understanding of string theory. The gauged supergravity theories have been studied for 25 years beginning with the first paper [1] which found an SO(8) gauged theory within the $D = 4$, $N = 8$ theory. These theories are sometimes called massive theories in that they are a deformation of the massless theory by a massive parameter. They have generally been found by starting from the massless supergravity theory in the dimension of interest and adding a deformation to the action such as a cosmological constant or a non-abelian interaction for the vectors and using supersymmetry closure to complete the theory. In relatively recent years all maximal gauged supergravity theories in each dimension D has been classified in terms of a single object called the embedding tensor which can be thought of as belonging to a representation of the internal symmetry group of the supergravity theory in D dimensions [2–8]. Thus for example all the gauged supergravity theories in five dimensions are parametrised modulo further constraints by an embedding tensor in the $\overline{\mathbf{351}}$ of the symmetry group E_6.

Certain gauged supergravities have played an important role in more recent developments. Two of the most important examples are the five dimensional gauged supergravity theory which results from dimensionally reducing the ten dimensional IIB supergravity theory on S^5 which is central to the AdS/CFT conjecture and those theories that occur in flux compactifications with a view to moduli stabilisation. However, it is fair to say that gauged supergravities in general have not been fitted into any conventional discussions of M theory.

It was conjectured in 2001 that the theory underlying string theory should possess an E_{11} symmetry and indeed the non-linear realisation of this symmetry contained the eleven dimensional supergravity theory [9]. By taking different decompositions of E_{11} one finds different supergravity theories. In particular, to find the theory in D dimensions one performs the decomposition of E_{11} into $GL(D,\mathbb{R}) \otimes G$ which corresponds to the algebra remaining after deleting the Dth node of the E_{11} Dynkin diagram. In particular in ten dimensions one finds two theories which have at low levels precisely the content of the IIA and IIB supergravity theories [9, 10]. Moreover, the Romans theory was found to be a non-linear realisation [11] which includes all form fields up to and including a 9-form with a corresponding set of generators. This 9-form is automatically encoded in the non-linear realisation of E_{11} [12], and its 10-form field-strength is dual to Romans cosmological constant.

More recently the from fields, that is those field with only completely anti-symmetrised Lorentz indices, were found in all dimensions, D [13, 14]. These include the $D-1$ forms in the D-dimensional theory whose equation of motion generically leads to a cosmological constant. As such the number of such forms should correspond to the number of gauged supergravity theories and indeed the representation of the $D-1$ forms is precisely the same as that of the embedding tensor used to classify the gauged supergravity theories. It was therefore apparent that E_{11} encoded all the possible maximal gauged supergravity theories. Thus for the first time the gauged supergravities were included in some underlying unifying formulation rather that found as the possible massive deformations in each dimension.

A feature that is always present in the E_{11} theories in different dimensions is that every form field has a corresponding dual field, indeed if the n-form fields belong to the representation $\mathbf{R_n}$ then we also find $D-n-2$-dual form fields in the complex conjugate representation, i.e. $\mathbf{R_{D-n-2}} = \overline{\mathbf{R}}_\mathbf{n}$. This was already apparent in the case of eleven dimensions and the IIA and IIB theories [9, 10]. As mentioned above the rank $D-1$ forms are dual to a cosmological constant while the rank D forms are not dual to anything but play an important role in brane associated dynamics. This can be thought of as a hierarchy of fields of ascending rank.

The results in the two paragraphs above are of a purely kinematical nature, however, progress has been made in constructing the dynamics of gauged supergravity theories using E_{11}. Initially this was achieved using the so called l_1 representation [15] to provide an E_{11} covariant generalised space-time [16]. While wishing to continue with this approach at a future date we also pursued an alternative more bottom up approach introducing only the usual D-dimensional space-time, with its corresponding space-time translations operator and at the same time extending the E_{11} algebra to include generators that had the effect of making local all the rigid Borel E_{11} transformations [17]. These so called Ogievetsky generators lead in the non-linear realisation to fields that can be eliminated covariantly and do not appear in the final dynamics. Nonetheless they play a crucial role in determining the field strengths of all the fields. Therefore, the algebra formed by the non-negative level E_{11} generators, the D-dimensional space-time translation generator and the above mentioned Ogievetsky generators, called $E_{11,D}^{\rm local}$ in [17], determines the field strengths of the massless maximal supergravity theories in any dimension. It also emerged in [17] that in the case of gauged supergravities the final dynamics is controlled by a massive deformation $\tilde{E}_{11,D}^{\rm local}$ of the algebra $E_{11,D}^{\rm local}$, in which the deformed E_{11} generators have a non-trivial commutation relation with the momentum operator. This was shown in detail for the case of the Scherk-Schwarz reduction of IIB to nine dimensions, the five-dimensional gauged maximal supergravity and Romans massive IIA theory. In the first case it was also shown that $\tilde{E}_{11,9}^{\rm local}$ is a subalgebra of the algebra $E_{11,10B}^{\rm local}$ that describes the IIB theory in ten dimensions, while the last case reproduced the results of [11], where the field strengths of the Romans theory were constructed adopting a non-trivial commutator between the E_{11} generators and momentum.

We note that gauged supergravities have also been discussed from the E_{10} [18] viewpoint. In particular the case of Romans IIA was discussed in [19] while the case of maximal gauged supergravity in three dimensions was analysed in [20].

In this paper we continue the analysis of [17] and construct all the massive deformations in each dimension. We find that the underlying E_{11} algebra and the Jacobi identities imply that the deformations in a given dimension are uniquely determined by one object that belongs to the same representation as the $D-1$ form generators and so can be identified with the embedding tensor used previously to classify gauged supergravity theories. We use the algebra to construct in a simple way all the fields strengths of all the gauged supergravities in all dimensions. The dynamics then arises as first order equations that are duality relations among these field strengths. In particular, the scalar equation results from the curl of the duality relation between the $D-2$-form fields and the scalars, using also the fact that the $D-1$-form field is dual to the embedding tensor. In general there is more than one gauge covariant quantity that one can construct contracting the scalars with the embedding tensor, and this procedure does not determine their relative coefficient, and therefore does not determine the exact form of the scalar potential. We analyse each dimension from three to nine, and these results, together with the ten-dimensional deformation corresponding to the Romans theory analysed in [17], give the field strengths of all possible massive maximal supergravities in any dimension.

The paper is organised as follows. In section 2 we derive the general method of constructing the deformed E_{11} algebra in any dimension. In section D, with $D = 3, \ldots, 9$, we explicitly derive the deformed algebra in a given dimension D. In section 10 we discuss the form of the duality relations that the various field strengths must satisfy in any dimension, and section 11 contains the conclusions. The paper also contains three appendices. In appendix A we review some group theoretic techniques and we derive from E_{11} the relevant projection formulae used in the paper. In appendix B we explicitly evaluate the field strengths in the general notation of section 2. In appendix C we derive the field strengths of the four-dimensional theory using a different method, that is based on the non-linear realisation of $E_{11} \otimes_s l_1$ and applies in four dimensions the analysis that was carried out in [16] in the five-dimensional case.

2 The general method

We wish to consider the formulation of the E_{11} algebra appropriate to D dimensions which can be found by decomposing E_{11} with respect to the algebra that results from deleting the Dth node of the E_{11} Dynkin diagram. This resulting algebra is $\mathrm{SL}(D, \mathbb{R}) \otimes G$ where $\mathrm{SL}(D, \mathbb{R})$ is associated with D-dimensional gravity and G is the internal symmetry algebra. The resulting form generators, that is those with only anti-symmetric Lorentz indices, are explicitly given in the later sections. In this section, we are interested in a universal treatment valid for every dimension and so we introduce a corresponding notation. We denote the form generators as $R^{a_1\ldots a_n, M_n}$ and the generators with no Lorentz indices are written as R^α. The latter generators are those of the internal symmetry algebra, G and the generators $R^{a_1\ldots a_n, M_n}$ carry the representation $\mathbf{R_n}$ of G which transforms the M_n indices. We note that in this notation $\mathbf{R_0}$ is the adjoint representation. For example, in the case of five dimensions $G = E_6$ and the form generators are given in eq. (5.1).

The E_{11} algebra involving the form generators is then given by

$$[R^{a_1\ldots a_m, M_m}, R^{b_1\ldots b_n, N_n}] = f^{M_m N_n}{}_{P_{n+m}} R^{a_1\ldots a_m b_1\ldots b_n, P_{n+m}} \tag{2.1}$$

and

$$[R^\alpha, R^{a_1\ldots a_m, M_m}] = (D^\alpha)_{N_m}{}^{M_m} R^{a_1\ldots a_m, N_m}, \quad [R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma \quad , \tag{2.2}$$

where $f^{M_m N_n}{}_{P_{n+m}}$ are generalised structure constants whose form will be shown in the following in several examples. By studying the table of [13] of forms contained in E_{11}, which is table 1 in this paper (observe that the table contains the representations of the fields, which are the complex conjugate of the representations of the corresponding generators), one finds that in all dimensions the representation of the 4-form generators is in the anti-symmetric tensor representation formed from two representations $\mathbf{R_2}$ and so it has indices $[M_2 N_2]$. As such we may write the 4-form generators as $R^{a_1 a_2 b_1 b_2, M_2 N_2} = R^{a_1 a_2 b_1 b_2, [M_2 N_2]}$;

in terms of our general notation the indices M_4 are for this generators represented by $[M_2N_2]$. As a result the commutator of two 2-form generators can be written as

$$[R^{a_1a_2M_2}, R^{b_1b_2N_2}] = R^{a_1a_2b_1b_2,M_2N_2} \quad , \tag{2.3}$$

where we have taken the constant of proportionality in the commutator to be one as this commutation relation can be taken to define the way the four form generator appears in the E_{11} algebra. The particularly simple form of this commutator will prove to be useful in this paper. Some other related observations that will be useful are that $\mathbf{R_1} \otimes \mathbf{R_n}$ contains the representation $\mathbf{R_{n+1}}$ and that $\mathbf{R_n} = \overline{\mathbf{R}}_{\mathbf{D-n-2}}$ for $n \neq D-1, D$. The first implies that one can find all form generators by taking repeated commutators of the one form generators and the second reflects that in the E_{11} formulation one finds dual fields for all the form fields usually associated with the physical degrees of freedom of the theory. Taking $n = 0$ we find that $\mathbf{R_{D-2}} = \mathbf{R_0}$ which is the adjoint representation and so it is real.

In fact the above algebra contains the E_{11} form generators that have positive level with respect to the level associated with node deletion discussed above. It is a truncation of the E_{11} algebra to contain just these generators. Clearly, eqs. (2.1) and (2.2) obey certain Jacobi identities which imply, for example, that the structure constants are invariant tensors of the internal symmetry group G. However, the structure constants also obey restrictions resulting from their E_{11} origin. These result from the Jacobi identities, but also from the construction of E_{11} from its Chevalley generators. In particular, the left hand side of the commutator of eq. (2.1) implies that the form generators on the right hand side must belong to the $\mathbf{R_n} \otimes \mathbf{R_m}$ representation of G, however, only the $\mathbf{R_{n+m}}$ representation arises. As a result, the structure constant $f^{M_mN_n}{}_{P_{n+m}}$ must obey the conditions that project onto only this latter representation. A particular example, that will be important for what follows, is the case for $m = 1$ and $n = D-2$ whose corresponding commutator has the form

$$[R^{a_1,N_1}, R^{a_2\dots a_{D-1},\alpha}] = f^{N_1\alpha}{}_{M_{D-1}} R^{a_1a_2\dots a_{D-1},M_{D-1}} \tag{2.4}$$

where the generator on the right hand side corresponds to the next to space-filling form fields that give rise in the non-linear realisation to the cosmological constant. Here we have used that $\mathbf{R_{D-2}}$ is the adjoint representation and so is labelled by $\alpha, \beta, \dots$. For the cases of $D = 4, 5, 6$ i.e. E_7 and E_6 and $E_5 \equiv D_5$ the $\mathbf{R_1} \otimes \mathbf{R}_{\rm adj}$ contains three irreducible representations, only one of which is the representation to which the next to space-filling generators belong. For the other dimensions one finds more representations in the tensor product, but in all cases there are two or more representations in the tensor product that must be projected out to find the representation, or for $D = 9, 8, 7, 3$ the two presentations to which the next to space-filling generators belong (see table 2). As such the structure constants $f^{N_1\alpha}{}_{N_{D-1}}$ must obey at least two projections conditions that turn out to be of the form

$$(D_\alpha)_{N_1}{}^{M_1} f^{N_1\alpha}{}_{N_{D-1}} = 0, \quad (D^\beta D_\alpha)_{N_1}{}^{M_1} f^{N_1\alpha}{}_{N_{D-1}} = c f^{M_1\beta}{}_{N_{D-1}} \tag{2.5}$$

for a suitable constant c. Such projector conditions are discussed in more detail in appendix A.

To the E_{11} algebra we add, as explained in reference [17], the space-time translation operator P_a and an infinite number of so called Ogievetsky generators. In fact for our purposes we need only add the lowest order such generators, $K^{a,b_1\ldots b_n,M_n}$, which by definition obey the commutator

$$[K^{a,b_1\ldots b_n,M_n},P_c] = \delta^a_c R^{b_1\ldots b_n,M_n} - \delta^{[a}_c R^{b_1\ldots b_n],M_n} \quad . \tag{2.6}$$

As this equation makes clear the generator $K^{a,b_1\ldots b_n,M_n}$ is associated with the E_{11} generators $R^{b_1\ldots b_n,M_n}$ and carries the same internal symmetry representation, $\mathbf{R_n}$. It also satisfies $K^{[a,b_1\ldots b_n],M_n}=0$. The Ogievetsky generators rotate into themselves under the action of the E_{11} generators and the commutator of two Ogievetsky generators gives another Ogievetsky generator. We take the space-time translation operator P_a to commute with the positive level generators of E_{11}. Indeed, it is this requirement that forces us to consider only the positive level generators of E_{11}.

We now consider a massive deformation of the above algebra which is parameterised by the symbol g and given by

$$\begin{aligned}
[R^{a_1\ldots a_m,M_m},R^{b_1\ldots b_n,M_n}] &= f^{M_mN_n}{}_{P_{n+m}}R^{a_1\ldots a_m b_1\ldots b_n,P_{n+m}} \\
&\quad + gL^{M_mN_n}{}_{P_{n+m-1}}K^{[a_1,a_2\ldots a_m]b_1\ldots b_n,P_{n+m-1}} &(2.7)\\
[R^\alpha,R^{a_1\ldots a_m,M_m}] &= (D^\alpha)_{N_m}{}^{M_m}R^{a_1\ldots a_m,N_m} &(2.8)\\
[R^{a_1\ldots a_m,M_m},P_c] &= -gW^{M_m}{}_{M_{m-1}}\delta^{[a_1}R^{a_2\ldots a_m],M_{m-1}} &(2.9)\\
[K^{a,b_1\ldots b_n,M_n},P_c] &= \delta^a_c R^{b_1\ldots b_n,M_n} - \delta^{[a}_c R^{b_1\ldots b_n],M_n} \\
&\quad + gU^{M_n}{}_{M_{n-1}}\delta^{[b_1}_c K^{|a|b_2\ldots b_n],M_{n-1}}, &(2.10)
\end{aligned}$$

while the deformation of eq. (2.3) for the commutator of two two forms is given by

$$[R^{a_1a_2M_2},R^{b_1b_2N_2}] = R^{a_1a_2b_1b_2,M_2N_2} + gV^{M_2N_2}{}_{P_3}K^{[a_1,a_2]b_1b_2,P_3} \quad . \tag{2.11}$$

For the case of the 4-form generator eq. (2.9) can be written as

$$[R^{a_1\ldots a_4,M_2N_2},P_c] = -gW^{M_2N_2}{}_{P_3}\delta^{[a_1}R^{a_2\ldots a_4],P_3} \quad . \tag{2.12}$$

The above commutators preserve the grading $[R]=0$, $[P]=-1$, $[K]=1$ provided we also assign $[g]=-1$ to the constant g. For each set of objects W we find a different deformation of the E_{11} algebra. The deformed algebra of eq. (2.7) to (2.12) is the general version of that given in [17] for special cases such as that for the the gauged nine-dimensional supergravity that arises from Scherk-Schwarz reduction of IIB, gauged five-dimensional maximal supergravity and Romans massive IIA.

We define $W^{N_1}_{Q_0}\equiv\Theta^{N_1}_{Q_0}=\Theta^{N_1}_\alpha$ as the index Q_0 is the index on R^{Q_0} which is just the index α. In terms of this notation the lowest order example of eq. (2.12) is given by

$$[R^{aN_1},P_b] = -g\delta^a_b\Theta^{N_1}_\alpha R^\alpha \quad . \tag{2.13}$$

We will see that $\Theta^{N_1}_\alpha$ will turn out to be the embedding tensor discussed so much in the literature on gauged supergravities.

The $\Theta_\alpha^{N_1}$, like all the objects W, are not invariant tensors of the internal symmetry group G. One can think of them as a kind of spurion; for each allowed value of $\Theta_\alpha^{N_1}$ one finds a different gauged supergravity, for example the local gauge group is determined by the value of $\Theta_\alpha^{N_1}$.

We will now work out the consequences of the Jacobi identities for the deformed algebra of eqs. (2.7) to (2.11). We begin with the Jacobi identities that arise from taking two E_{11} generators and P_c. These will place linear conditions on $W^{P_{n+1}}{}_{S_n}$ as we have only one P_c. In particular we first consider the identity

$$[[R^{a_1M_1},R^{b_1\dots b_n,N_n}],P_c]=[R^{a_1M_1},[R^{b_1\dots b_n,N_n},P_c]]+[[R^{a_1M_1},P_c],R^{b_1\dots b_n,N_n}] \quad . \tag{2.14}$$

We evaluate this using eqs. (2.7) to (2.13). Not all structures of Lorentz indices that arise are independent due to the identity

$$n\delta_c^{[b_1|}R^{a|b_2\dots b_n]}=\delta_c^aR^{b_1b_2\dots b_n}-(n+1)\delta_c^{[a}R^{b_1b_2\dots b_n]} \quad . \tag{2.15}$$

As such it suffices to consider the coefficients of only the terms involving $\delta_c^{[a}R^{b_1b_2\dots b_n]}$ and those of the form $\delta_c^aR^{b_1b_2\dots b_n}$ and use the above equation to express any other contributions in terms of these two forms. We find that at order g this leads, respectively, to the two equations

$$f^{M_1N_n}{}_{P_{n+1}}W^{P_{n+1}}{}_{S_n}=X_n{}^{M_1}{}_{S_n}{}^{N_n}-f^{M_1Q_{n-1}}{}_{S_n}W^{N_n}{}_{Q_{n-1}} \tag{2.16}$$

and

$$L^{M_1N_n}{}_{P_n}=-\frac{1}{n}f^{M_1Q_{n-1}}{}_{P_n}W^{N_n}{}_{P_n}-X_n{}^{M_1}{}_{P_n}{}^{N_n} \tag{2.17}$$

where $X_n{}^{M_1}{}_{S_n}{}^{N_n}=\Theta_\alpha^{M_1}(D^\alpha)_{S_n}{}^{N_n}$.

As noted above, the representation $\mathbf{R_1}\otimes\mathbf{R_n}$ always contains the representation $\mathbf{R_{n+1}}$ and so the structure constant $f^{M_1N_n}{}_{P_{n+1}}$ can be inverted to leave only $W^{P_{n+1}}{}_{S_n}$ on the right-hand side of eq. (2.14). Thus this equation solves for $W^{P_{n+1}}{}_{S_n}$ in terms of $\Theta_\alpha^{M_1}$ and the lower level $W^{P_n}{}_{S_{n-1}}$ and these equations provide a set of recursion relations that allow one to solve for all the $W^{P_{n+1}}{}_{S_n}$'s in terms of $\Theta_\alpha^{M_1}$. Eq. (2.12) then just gives $L^{M_1N_n}{}_{P_n}$ in terms of $\Theta_\alpha^{M_1}$.

At order g^2 we find that the Jacobi identity of eq. (2.16) implies the relation

$$L^{M_1N_n}{}_{P_n}U^{P_n}{}_{S_{n-1}}=L^{M_1Q_{n-1}}{}_{S_{n-1}}W^{N_n}{}_{Q_{n-1}} \quad . \tag{2.18}$$

At lowest order eq. (2.14) implies that

$$f^{M_1N_1}{}_{P_2}W^{P_2}{}_{S_1}=2X_1{}^{(M_1}{}_{S_1}{}^{N_1)} \quad . \tag{2.19}$$

In deriving this relation we have used that

$$[R^{aM_1},\Theta_\alpha^{N_1}R^\alpha]=f^{M_1Q_0}{}_{S_1}W^{N_1}{}_{Q_0}R^{aS_1}=-X_1^{N_1}{}_{S_1}{}^{M_1}R^{aS_1} \tag{2.20}$$

since in terms of our notation $f^{M_1Q_0}{}_{S_1}=-f^{Q_0M_1}{}_{S_1}=-(D^\alpha)_{S_1}{}^{M_1}$ and using our earlier definition $\Theta_\alpha^{N_1}=W^{N_1}{}_{Q_0}$.

While we have solved for all the W's in terms of Θ using the above equations it is more practical to do this step for the W involved with the 4-form generator using the Jacobi identity

$$[[R^{a_1a_2M_2}, R^{b_1b_2,N_2}], P_c] = [R^{a_1a_2M_2}, [R^{b_1b_2,N_2}, P_c]] + [[R^{a_1a_2M_2}, P_c], R^{b_1b_2,N_2}] \tag{2.21}$$

and eqs. (2.11) and (2.12) rather than the Jacobi identity of eq. (2.14) for the case of $m = 1$ and $n = 3$. Using similar arguments to those deployed above, we find at order g^1 the two equations

$$W^{M_2N_2}{}_{R_3} = -W^{N_2}{}_{R_1} f^{R_1M_2}{}_{R_3} + W^{M_2}{}_{R_1} f^{R_1N_2}{}_{R_3} \tag{2.22}$$

and

$$V^{M_2N_2}{}_{R_3} = W^{M_2N_2}{}_{R_3} \quad . \tag{2.23}$$

Clearly, these solve for $W^{M_2N_2}{}_{R_3}$ and $V^{M_2N_2}{}_{R_3}$ in terms of $W^{N_2}{}_{R_1}$ and so in terms of $\Theta_\alpha^{N_1} R^\alpha$, while at order g^2 we find that eq. (2.21) implies that

$$\frac{1}{3} V^{M_2N_2}{}_{P_3} U^{P_3}{}_{P_2} = -L^{R_1M_2}{}_{P_2} W^{N_2}_{R_1} \quad . \tag{2.24}$$

It will be useful to also consider the Jacobi identity

$$[\Theta_\alpha^{N_1} R^\alpha, [R^{a_1\ldots a_n,P_n}, P_c]] = [\Theta_\alpha^{N_1} R^\alpha, R^{a_1\ldots a_n,P_n}], P_c] \tag{2.25}$$

since $[R^\alpha, P_c] = 0$. It implies that

$$X^{N_1}_{n-1}{}_{R_{n-1}}{}^{Q_{n-1}} W^{P_n}{}_{Q_{n-1}} = X^{N_1}_{n}{}_{Q_n}{}^{P_n} W^{Q_n}{}_{R_{n-1}} \quad . \tag{2.26}$$

We now consider the consequences of the Jacobi identities that involve one E_{11} generator $R^{a_1\ldots a_n,P_n}$ and the generators P_c and P_d. This implies a quadratic constraint on $W^{P_n}{}_{Q_{n-1}}$'s that is given by

$$W^{P_n}{}_{Q_{n-1}} W^{Q_{n-1}}{}_{R_{n-2}} = 0 \quad . \tag{2.27}$$

At the lowest order, i.e. $n = 1$, eq. (2.26) implies that

$$X^{N_1}_{1}{}_{Q_1}{}^{P_1} \Theta^{Q_1}_\alpha = \Theta^{N_1}_\epsilon f^{\epsilon\gamma}{}_\alpha \Theta^{P_1}_\gamma \quad . \tag{2.28}$$

Finally we consider the Jacobi identity with $K^{a,b_1\ldots b_n,M_n}$ and P_c and P_d, namely

$$[[K^{a,b_1\ldots b_n,M_n}, P_c], P_d] - [[K^{a,b_1\ldots b_n,M_n}, P_d], P_c] = 0 \quad , \tag{2.29}$$

as $[P_c, P_d] = 0$. At order g^1 we find that

$$U^{P_n}{}_{P_{n-1}} = -\frac{n}{n+1} W^{P_n}{}_{P_{n-1}} \quad , \tag{2.30}$$

while at order g^2 we find that

$$U^{P_n}{}_{P_{n-1}} U^{P_{n-1}}{}_{P_{n-2}} = 0 \quad . \tag{2.31}$$

The first equation solves for $U^{P_n}{}_{P_{n-1}}$ in terms of $W^{P_n}{}_{P_{n-1}}$ and so in terms of $\Theta_\alpha^{N_1}$. Using eq. (2.28) and eq. (2.30) we observe that eq. (2.31) is automatically solved. Furthermore substituting eq. (2.30) into eq. (2.24) we find it is automatically satisfied using eq. (2.17).

We now summarise the content of this section so far. The deformation of the algebra of eqs. (2.7) to (2.12) involves a number of the constants, namely $L^{M_m N_n}{}_{P_{n+m-1}}$, $W^{M_m}{}_{M_{m-1}}$, $U^{P_n}{}_{P_{n-1}}$, $V^{M_2 N_2}{}_{P_3}$ and $W^{M_2 N_2}{}_{P_3}$. However, the Jacobi identities imply that all of these may be solved in terms of the W's and these are in turn determined in terms of the single object $\Theta_\alpha^{N_1}$. *Thus the entire deformation is determined in terms of $\Theta_\alpha^{N_1}$, or equivalently eq. (2.13).*

However, the above equations also impose constraints on $\Theta_\alpha^{N_1}$. Clearly, there are the quadratic constraints of eq. (2.28) which are a set of constraints on $\Theta_\alpha^{N_1}$ once we have substituted for the W's in terms of $\Theta_\alpha^{N_1}$. However, we also have a set of linear constraints that originate from eq. (2.16) whose right hand side can be expressed entirely in terms of $\Theta_\alpha^{N_1}$, a variable in which it is linear. As explained above the structure constant that occurs in the commutator of eq. (2.1) obeys projector conditions arising from the fact that the form generators on the right hand side do not belong to the representation $\mathbf{R_m} \otimes \mathbf{R_n}$, but only to the representation $\mathbf{R_{n+m}}$ that it contains. The number of projection conditions correspond to the number of irreducible representation in $\mathbf{R_m} \otimes \mathbf{R_n}$ which are not contained in the representation $\mathbf{R_{n+m}}$. However, certain of these structure constants, i.e. $f^{M_1 N_n}{}_{P_{n+1}}$ appear on the left hand side of eq. (2.16) and so the object $\Theta_\alpha^{N_1}$ that appears on the right hand side of this equation will satisfy corresponding constraints. In particular, taking $n = D-2$ in eq. (2.16) we find the structure constant $f^{M_1 \alpha}{}_{P_{D-1}}$ on the left hand side which obeys the constraints of eq. (2.5) for the cases of dimensions four, five and six. This is evident from table 2 where we find that in these dimensions the representation $\mathbf{R_1} \otimes \mathbf{R_0}$ contains three irreducible representations only one of which is $\mathbf{R_{D-1}}$. As explained in appendix A this is a consequence of the fact that for these dimensions $\mathbf{R_1}$ is the fundamental representation of the internal symmetry group. In other dimensions one has to project out more than two irreducible representations from $\mathbf{R_1} \otimes \mathbf{R_0}$ to leave the representation $\mathbf{R_{D-1}}$ (see table 2) and so one has more projection conditions on the structure constant $f^{M_1 \alpha}{}_{P_{D-1}}$ and so on $\Theta_\alpha^{N_1}$. Thus in dimensions four, five and six we will find two linear conditions on $\Theta_\alpha^{N_1}$ which are evaluated in detail later in this paper and are found to be

$$(D^\alpha)_{N_1}{}^{M_1} \Theta_\alpha^{N_1} = 0 \tag{2.32}$$

and

$$(D_\beta D^\alpha)_{N_1}{}^{M_1} \Theta_\alpha^{N_1} = c \Theta_\beta^{M_1} \tag{2.33}$$

where c is a constant plus possible further constraints. In dimension other than four, five and six we find these constraints as well as further constraints. However, in dimensions other than three, four, five and six one finds that all the conditions on $\Theta_\alpha^{N_1}$ already arise at lower levels than $n = D-2$ from similar conditions on the corresponding lower level structure constants. Hence, a priori although $\Theta_\alpha^{N_1}$ could belong to the representation $\mathbf{R_1} \otimes \mathbf{R_0}$ the constraints discussed above, and derived in detail in each dimension in later sections, restrict it to actually belong to the same representation as the $D-1$ form generators i.e. the $\mathbf{R_{D-1}}$ representation in all dimensions.

To summarise this section so far. *We have found that the deformation is uniquely determined in terms of* $\Theta_\alpha^{N_1}$ *and will find, taking account of results in later sections, that this object obeys constraints that imply that it belongs to the same representation as the* $D-1$ *form generators.*

We turn our attention to the construction of the field strengths from the Cartan forms. We write the group element of the algebra of eqs. (2.7) to (2.12) in the form

$$g = e^{x^a P_a} e^{\Phi \cdot K} e^{A \cdot R} \tag{2.34}$$

where

$$e^{A \cdot R} = \ldots e^{A_{a_1 \ldots a_m, M_m} R^{a_1 \ldots a_m, M_m}} e^{A_{a_1 \ldots a_{m-1}, M_{m-1}} R^{a_1 \ldots a_{m-1}, M_{m-1}}} \ldots \\ e^{A_{a_1 a_2, M_2} R^{a_1 a_2, M_2}} e^{A_{a_1, M_1} R^{a_1, M_1}} g_\varphi \tag{2.35}$$

where $g_\varphi = e^{\varphi_\alpha R^\alpha}$ and $e^{\Phi \cdot K}$ is a similar expression involving the Ogievetsky fields and generators. The field strengths are contained in the Cartan forms which we can write as

$$g^{-1} dg = \mathcal{V}^{(0)} + \mathcal{V}^{(1)} + \ldots \tag{2.36}$$

where $\mathcal{V}^{(n)}$ is the contribution at g^n. The full calculation involves many terms but we are only interested in the field strengths and so we will only keep terms that contain E_{11} generators. These contain terms of the form $dx^\mu G_{\mu a_1 \ldots a_n, M_n} R^{a_1 \ldots a_n, M_n}$. The coefficients $G_{\mu a_1 \ldots a_n, M_n}$ are not totally anti-symmetrised in all their $\mu a_1 \ldots a_n$ indices, but the terms that are not are set to zero using the inverse Higgs mechanism which solves for the corresponding Ogievetsky field. This mechanism is discussed in detail in reference [17]. The term that is totally anti-symmetrised is the field strength and as this is what is needed for the dynamics we will compute only this term. To carry out this task we only need the commutation relation of eqs. (2.7) and (2.9) and need not include the Ogievetsky fields in our computations.

Let us denote the totally anti-symmetric part of $\mathcal{V}$ by $\mathcal{V}_A$ and write it as

$$\mathcal{V}_A = \sum_m \frac{1}{m+1} \tilde{F}_{\mu a_1 \ldots a_m, M_m} R^{a_1 \ldots a_m, M_m} = g_\varphi^{-1} \sum_m \frac{1}{m+1} F_{\mu a_1 \ldots a_m, M_m} R^{a_1 \ldots a_m, M_m} g_\varphi \tag{2.37}$$

where $F_{\mu a_1 \ldots a_m, M_m} = F_{[\mu a_1 \ldots a_m], M_m}$. We denote the order g^p contribution by $F^{(p)}_{\mu a_1 \ldots a_m, M_m}$, and the structure of the algebra is such that only the order zero and the first order in g occur. The factors of g_φ lead to the matrix functions $(e^{-\varphi_\alpha D^\alpha})_{M_n}{}^{N_n}$ where $(D^\alpha)_{M_n}{}^{N_n}$ is in the corresponding representation $\mathbf{R_m}$. That is $\tilde{F}_{\mu a_1 \ldots a_m, M_m} = (e^{-\varphi_\alpha D^\alpha})_{M_m}{}^{N_m} F_{\mu a_1 \ldots a_m, N_m}$. As these extra factors involving the scalars just complicate the formulae we will only explicitly compute the $F_{\mu a_1 \ldots a_m, M_m}$. The scalar factor just converts $F_{\mu a_1 \ldots a_m, M_m}$ which is in the linear representation $\mathbf{R_m}$ into $\tilde{F}_{\mu a_1 \ldots a_m, M_m}$ which is in a non-linear representation of the internal symmetry, in fact transforming by a non-linear local subgroup rotation.

The terms in $\mathcal{V}_A^{(0)}$ are just the field strengths found from the E_{11} algebra without any deformation and these have been computed in several cases before. We will begin by computing these terms for any dimension making use of the notation developed above. We use the well known relation

$$e^{-A}de^{A}=\frac{1-e^{-A}}{A}\star dA \quad , \tag{2.38}$$

where A is a generic operator and the $\star$-product is defined by

$$A\star B=[A,B] \qquad , \qquad A^{p}\star B=[A,A^{p-1}\star B] \quad . \tag{2.39}$$

We note that

$$[A_{a_1\ldots a_m,M_m}R^{a_1\ldots a_m,M_m},R^{b_1\ldots b_n,N_n}]=L_{a_1\ldots a_m\,P_{m+n}}{}^{N_n}R^{a_1\ldots a_m b_1\ldots b_n,P_{m+n}} \quad , \tag{2.40}$$

where

$$L_{a_1\ldots a_m\,P_{m+n}}{}^{N_n}=A_{a_1\ldots a_m,M_m}f^{M_mN_n}{}_{P_{m+n}} \quad . \tag{2.41}$$

Using these conventions we may also evaluate

$$\begin{aligned}&A_{a_1\ldots a_m,M_m}R^{a_1\ldots a_m,M_m}\star A_{b_1\ldots b_n,M_n}R^{b_1\ldots b_n,M_n}\star R^{c_1\ldots c_p,P_p}\\&\qquad=\{L_{a_1\ldots a_m}L_{b_1\ldots b_n}\}_{P_{m+n+p}}{}^{P_p}R^{a_1\ldots a_m b_1\ldots b_n,c_1\ldots c_p P_{m+n+p}} \quad ,\end{aligned} \tag{2.42}$$

where the two L factors are multiplied using matrix multiplication on their internal symmetry indices.

Denoting with

$$g_A^m=e^{A_{a_1\ldots a_m,M_m}R^{a_1\ldots a_m,M_m}} \tag{2.43}$$

and

$$g_A^{<m}=e^{A_{a_1\ldots a_{m-1},M_{m-1}}R^{a_1\ldots a_{m-1},M_{m-1}}}\ldots e^{A_{a_1a_2,M_2}R^{a_1a_2,M_2}}e^{A_{a_1,M_1}R^{a_1,M_1}}g_\varphi \quad , \tag{2.44}$$

we write

$$\mathcal{V}_A^{(0)}=\sum_m (g_A^{<m})^{-1}(g_A^m)^{-1}dg_A^m\ g_A^{<m} \quad . \tag{2.45}$$

This expression can be evaluated using eq. (2.38). The result is further simplified by using the language of forms. We find that

$$\begin{aligned}F^{(0)}_{\mu a_1\ldots a_m,N_m}dx^\mu\wedge dx^{a_1}\wedge\ldots\wedge dx^{a_m}&=(m+1)\sum_{n_1,\ldots,n_r}\frac{(-1)^{n_1}}{n_1!}\cdots\frac{(-1)^{n_{r-1}}}{n_{r-1}!}\frac{(-1)^{n_r}}{(n_r+1)!}\\&dx^\mu\wedge\{(L^{(1)}\wedge\ldots\wedge L^{(1)})(L^{(2)}\wedge\ldots\wedge L^{(2)})\ldots(L^{(r)}\wedge\ldots\wedge L^{(r)})\}_{N_m}{}^{N_r}\partial_\mu A_{N_r},\end{aligned} \tag{2.46}$$

where $\partial_\mu A_{N_r}=\partial_\mu A_{a_1\ldots a_r,N_r}dx^{a_1}\wedge\ldots\wedge dx^{a_r}$ and $L^{(m)}{}_\bullet{}^\bullet=A_{a_1\ldots a_m,M_m}dx^{a_1}\wedge\ldots\wedge dx^{a_m}f^{M_m\bullet}{}_\bullet$. The sum being over all integers n_p such that $m=n_1+2n_2+\ldots+(r-1)n_{r-1}+r(n_r+1)$.

We now compute the analogous terms at order g^1, that is those involving totally anti-symmetric indices and E_{11} generators. Using eq. (2.9) we find that

$$\begin{aligned}\mathcal{V}_A^{(1)} &= e^{-A\cdot R} dx^\mu P_\mu e^{A\cdot R} \\ &= g\sum_m (g_A^{<m})^{-1} \frac{1-(g_A^m)^{-1}}{A_{a_1\ldots a_m,M_m} R^{a_1\ldots a_m,M_m}} \times \\ &\qquad \times W^{N_m}{}_{N_{m-1}} dx^\mu A_{\mu a_1\ldots a_{m-1},N_m} R^{a_1\ldots a_{m-1},N_{m-1}}\; g_A^{<m} + dx^\mu e_\mu{}^a P_a \quad .\end{aligned} \tag{2.47}$$

Further evaluating this expression using the above notation we find that

$$\begin{aligned}F^{(1)}_{\mu a_1\ldots a_m,N_m} dx^\mu \wedge dx^{a_1}\ldots\wedge dx^{a_m} &= (m+1)g\Bigg(\sum_{n_1,\ldots,n_r} \frac{(-1)^{n_1}}{n_1!}\cdots\frac{(-1)^{n_{r-1}}}{n_{r-1}!}\frac{(-1)^{n_r}}{(n_r+1)!} \\ dx^\mu \wedge \{(L^{(1)}\wedge\ldots\wedge L^{(1)})(L^{(2)}\wedge\ldots\wedge L^{(2)})&\ldots(L^{(r)}\wedge\ldots\wedge L^{(r)})\}_{N_m}{}^{N_{r-1}} W^{N_r}{}_{N_{r-1}} A^{(r)}_{\mu N_r} \\ -\frac{(-1)^m}{m+1!} dx^\mu \wedge \{L^{(1)}\wedge\ldots\wedge L^{(1)}\}_{N_m}{}^{R_1} X_1^{N_1}{}_{R_1}{}^{M_1} A^{(1)}_{M_1} A^{(1)}_{\mu N_1}\Bigg) \quad ,\end{aligned} \tag{2.48}$$

where $A^{(r)}_{\mu,N_r} = A_{\mu a_1\ldots a_{r-1},N_r} dx^{a_1}\wedge\ldots\wedge dx^{a_{r-1}}$ and $A^{(1)}_{M_1} = A_{a,M_1} dx^a$. The sum is such that $m = n_1 + 2n_2 + \ldots + (r-1)(n_{r-1}+1) + rn_r$, there being n_p factors of $L^{(p)}$ in the first term, where r must be greater than 1, and $m-1$ factors of $L^{(1)}$ in the second term. The contribution to the one form field strength consists of the term $gA_{\mu M_1} dx^\mu \Theta_\alpha^{M_1} R^\alpha$.

To summarise this section. *We have found all deformations of the form of eqs. (2.7) to (2.12) are determined by the one variable $\Theta_\alpha^{N_1}$ and this belongs to the same representation as the $D-1$ forms, i.e.* $\mathbf{R_{D-1}}$*, as well as satisfying certain quadratic constraints. We have computed the fields strengths that occur in the non-linear realisation of the deformed algebra. Thus we have found all the field strengths of all the maximal supergravities in all dimensions.* We have therefore reduced the computation of the field strengths and gauge transformations of gauged supergravities to a purely algebraic construction based on E_{11}.

To conclude this section, we discuss the gauge transformations of the fields. These arise in the non-linear realisation as rigid transformations of the group element, $g \to g_0 g$, as long as one includes the Ogievetsky generators [17]. In particular, in the massless theory the action of

$$g_0 = \exp(a_{a_1\ldots a_n,N_n} R^{a_1\ldots a_n,N_n}) \tag{2.49}$$

generates a global transformation of the fields of parameter $a_{a_1\ldots a_n,N_n}$, and the net effect of including the Ogievetsky generators is to promote this global symmetry to a local one via the identification

$$a_{a_1\ldots a_n,N_n} \to \partial_{[a_1}\Lambda_{a_2\ldots a_n],N_n} \quad . \tag{2.50}$$

In the massive theory, this is modified due to the fact that the E_{11} generators have non-trivial commutation relations with the momentum operator. If one acts with g_0 as in eq. (2.49) on the group element of eq. (2.34) and uses eq. (2.9), passing through $e^{x^a P_a}$ generates the term

$$\exp(-gW^{N_n}{}_{N_{n-1}} x^{a_1} a_{a_1\ldots a_n,N_n} R^{a_2\ldots a_n,N_{n-1}}) \quad . \tag{2.51}$$

Therefore, together with the constant transformation generated by the action of the term in eq. (2.49), the massive theory develops a transformation that is linear in x. The inclusion of the Og generators then has the net effect of promoting $a_{a_1 \dots a_n, N_n}$ to a local parameter, and the gauge transformation of the fields is obtained by taking the global $a_{a_1 \dots a_n, N_n}$ of the massless theory and making the identification

$$a_{a_1 \dots a_n, N_n} \to \partial_{[a_1} \Lambda_{a_2 \dots a_n], N_n} - g W^{N_{n+1}}{}_{N_n} \Lambda_{a_1 \dots . a_n, N_{n+1}} \tag{2.52}$$

instead of that of eq. (2.50). Indeed taking $\Lambda_{a_1 \dots a_{n-1}, N_n}$ to be at most linear in x this identification reproduces the transformations generated by eqs. (2.49) and (2.51). Therefore the gauge transformations of all the fields in the massive theory are given by the ones of the massless theory, provided that one makes the change

$$\partial_{[a_1} \Lambda_{a_2 \dots a_n], N_n} \to \partial_{[a_1} \Lambda_{a_2 \dots a_n], N_n} - g W^{N_{n+1}}{}_{N_n} \Lambda_{a_1 \dots . a_n, N_{n+1}} \quad . \tag{2.53}$$

A special case is the case $n = 0$ in eq. (2.52), for which although the first term on the right-hand side is not present, the second term gives a gauge transformation of parameter

$$- g \Theta_\alpha^{M_1} \Lambda_{M_1} \quad . \tag{2.54}$$

This determines the way in which all the fields transform under the gauge parameter Λ_{M_1} at order g, the field $A_{a_1 \dots a_n, N_n}$ transforming as

$$\delta A_{a_1 \dots a_n, N_n} = - g \Theta_\alpha^{M_1} \Lambda_{M_1} D^\alpha_{N_n}{}^{M_n} A_{a_1 \dots a_n, M_n} \quad . \tag{2.55}$$

In sections from 3 to 9 we will apply the results of this section to all dimensions from 3 to 9, showing that in all cases $\Theta_\alpha^{M_1}$ and the $D-1$ forms belong to the same representation and determining the field strengths and gauge transformations of all the form fields of any maximal supergravity theory in any dimension.

3 D=3

The bosonic sector of the massless maximal supergravity theory in three dimensions [21] describes 128 scalars parametrising the manifold $E_{8(+8)}/\mathrm{SO}(16)$ and the metric. This theory arises from the E_{11} decomposition appropriate to three dimensions, corresponding to the deletion of node 3 as shown in the Dynkin diagram of figure 1. The 1-form generators of E_{11} that arise in this decomposition belong to the **248** of E_8. The corresponding fields are dual to the scalars. There are also 2-form generators in the $\mathbf{1} \oplus \mathbf{3875}$, the corresponding fields having vanishing field-strength in the massless theory. We will not consider in our analysis the 3-forms and all the generators with mixed symmetry. To summarise, we consider the form generators

$$R^\alpha \quad (\mathbf{248}) \qquad R^{a,\alpha} \quad (\mathbf{248}) \qquad R^{a_1 a_2, M} \quad (\mathbf{3875}) \qquad R^{a_1 a_2} \quad (\mathbf{1}) \quad , \tag{3.1}$$

where $\alpha = 1, \dots, 248$ denotes the adjoint and $M = 1, \dots, 3875$ denotes the **3875** of E_8.

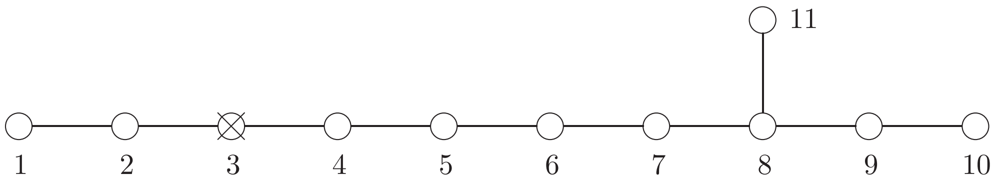

Figure 1. The E_{11} Dynkin diagram corresponding to 3-dimensional supergravity. The internal symmetry group is $E_{8(+8)}$.

The E_{11} commutation relations involving the generators in eq. (3.1) are

$$\begin{aligned}
[R^\alpha, R^\beta] &= f^{\alpha\beta}{}_\gamma R^\gamma \\
[R^\alpha, R^{a,\beta}] &= f^{\alpha\beta}{}_\gamma R^{a,\gamma} \\
[R^\alpha, R^{a_1 a_2}] &= 0 \\
[R^\alpha, R^{a_1 a_2,M}] &= D^{\alpha}_{N}{}^{M} R^{a_1 a_2,N} \\
[R^{a_1,\alpha}, R^{a_2,\beta}] &= g^{\alpha\beta} R^{a_1 a_2} + S^{\alpha\beta}_M R^{a_1 a_2,M} \quad ,
\end{aligned} \tag{3.2}$$

where $g^{\alpha\beta}$ is proportional to the Cartan-Killing metric and it is the metric we use to raise E_8 indices in the adjoint, $D^{\alpha}_{N}{}^{M}$ are the E_8 generators in the **3875** and $S^{\alpha\beta}_M$ is an E_8 invariant tensor. This invariant tensor is such that $S^{\alpha\beta}_M R^{a_1 a_2,M}$ belongs to the **3875**, and using the E_8 conventions and projection formulae of [22] one deduces that $S^{\alpha\beta}_M$ must satisfy the further conditions

$$\begin{aligned}
g_{\alpha\beta} S^{\alpha\beta}_M &= 0 \\
S^{\alpha\beta}_M &= -\frac{1}{12} f^{\epsilon}{}_{\gamma}{}^{\alpha} f_{\epsilon\delta}{}^{\beta} S^{\gamma\delta}_M \quad .
\end{aligned} \tag{3.3}$$

Indeed $S^{\alpha\beta}_M$ is symmetric in $\alpha\beta$, and the symmetric product of two **248** representations is

$$[\mathbf{248} \otimes \mathbf{248}]_{\rm S} = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000} \quad . \tag{3.4}$$

The conditions of eq. (3.3) project out $\mathbf{1} \oplus \mathbf{27000}$ to ensure that $S^{\alpha\beta}_M R^{a_1 a_2,M}$ belongs to the **3875**. The E_8 metric is related to the structure constant by [22]

$$g^{\alpha\beta} = -\frac{1}{60} f^{\alpha}{}_{\gamma\delta} f^{\beta\gamma\delta} \quad , \tag{3.5}$$

while another useful E_8 identity is [22]

$$f_{\alpha\epsilon\tau} f_{\beta}{}^{\epsilon}{}_{\sigma} f^{\gamma}{}_{\rho}{}^{\tau} f^{\delta\rho\sigma} = 24 \delta^{\gamma}_{(\alpha} \delta^{\delta}_{\beta)} + 12 g_{\alpha\beta} g^{\gamma\delta} - 20 f^{\epsilon}{}_{\alpha}{}^{\gamma} f_{\epsilon\beta}{}^{\delta} + 10 f^{\epsilon}{}_{\alpha}{}^{\delta} f_{\epsilon\beta}{}^{\gamma} \quad . \tag{3.6}$$

From the group element

$$g = e^{x\cdot P} e^{A_{a_1 a_2} R^{a_1 a_2}} e^{A_{a_1 a_2,M} R^{a_1 a_2,M}} e^{A_{a,\alpha} R^{a,\alpha}} e^{\phi_\alpha R^\alpha} \tag{3.7}$$

one derives the field-strengths of the 1-forms and 2-forms. These indeed result from anti-symmetrising the various terms in the Maurer-Cartan form, which is computed imposing that the generators in eq. (3.1) commute with momentum. We now consider the deformations of the algebra of eq. (3.2) resulting from imposing that the generators have a non-trivial commutation relation with momentum compatibly with the Jacobi identities.

We consider the general analysis of the previous section, applied to the three-dimensional case via the identifications

$$\begin{aligned} R^{a_1,M_1} &\to R^{a_1,\alpha} \\ R^{a_1a_2,M_2} &\to R^{a_1a_2}\,, \quad R^{a_1a_2,M} \\ \Theta^{M_1}{}_{\alpha} &\to \Theta^{\beta}{}_{\alpha} \\ W^{M_2}{}_{M_1} &\to W_{\alpha}\,, \quad W^{M}{}_{\alpha} \quad . \end{aligned} \tag{3.8}$$

Eq. (2.19), resulting from the Jacobi identity of two 1-forms and momentum, reads

$$W^M{}_\delta S^{\alpha\beta}_M + W_\delta g^{\alpha\beta} = \Theta^\alpha{}_\gamma f^{\gamma\beta}{}_\delta + \Theta^\beta{}_\gamma f^{\gamma\alpha}{}_\delta \quad . \tag{3.9}$$

The embedding tensor $\Theta^\alpha{}_\beta$ has no a priori symmetry, and thus is in the representations generated by the symmetric product of two **248** given in eq. (3.4) together with those generated in the antisymmetric product

$$[\mathbf{248} \otimes \mathbf{248}]_{\mathrm{A}} = \mathbf{248} \oplus \mathbf{30380} \quad . \tag{3.10}$$

We now show that eq. (3.9) rules out the possibility that the embedding tensor is antisymmetric. Using eq. (3.3) one derives from eq. (3.9) the condition

$$\Theta^\alpha{}_\gamma f^{\gamma\beta}{}_\delta + \Theta^\beta{}_\gamma f^{\gamma\alpha}{}_\delta - \frac{1}{31} g^{\alpha\beta}\Theta_{\gamma\rho} f^{\gamma\rho}{}_\delta + \frac{1}{12}\Theta^\gamma{}_\sigma f^{\sigma\rho}{}_\delta [f^\epsilon{}_\gamma{}^\alpha f_{\epsilon\rho}{}^\beta + f^\epsilon{}_\rho{}^\alpha f_{\epsilon\gamma}{}^\beta] = 0 \,. \tag{3.11}$$

Taking Θ antisymmetric and contracting β and δ this equation gives

$$f^{\alpha\beta\gamma}\Theta_{\beta\gamma} = 0 \quad , \tag{3.12}$$

which rules out the **248**. Using this and contracting eq. (3.11) with $f_{\tau\beta}{}^\delta$ one then shows that the antisymmetric part of Θ vanishes completely, thus ruling out the **30380** too. The fact that the **248** is ruled out also implies

$$W_\alpha = 0 \quad , \tag{3.13}$$

as can be seen contracting α and β in eq. (3.9).

We thus take Θ to be symmetric, which corresponds to the representations in eq. (3.4). The tensor $W^M{}_\alpha$ has indices in $\mathbf{3875} \otimes \mathbf{248}$, and this leads to the irreducible representations

$$\mathbf{3875} \otimes \mathbf{248} = \mathbf{779247} \oplus \mathbf{147250} \oplus \mathbf{30380} \oplus \mathbf{3875} \oplus \mathbf{248} \quad . \tag{3.14}$$

Therefore $W^M{}_\alpha$ is not along the **27000**. From eq. (3.9) it then follows that taking Θ to be in the **27000** one gets

$$(\Theta^\alpha{}_\gamma)_{\mathbf{27000}} f^{\gamma\beta}{}_\delta + (\Theta^\beta{}_\gamma)_{\mathbf{27000}} f^{\gamma\alpha}{}_\delta = 0 \quad , \tag{3.15}$$

which is inconsistent because it is the condition of invariance of Θ. Therefore the **27000** is also ruled out. The invariant tensor $S^{\alpha\beta}_M$ satisfies

$$S^{\alpha\beta}_M S_{\alpha\beta N} = \delta_{MN} \quad , \tag{3.16}$$

where δ_{MN} is the invariant tensor in the product $[\mathbf{3875} \otimes \mathbf{3875}]_{\rm S}$, and using this and eq. (3.13) one can invert eq. (3.9) to get

$$W^M{}_\delta = 2\Theta^\alpha{}_\gamma f^{\gamma\beta}{}_\delta S^M_{\alpha\beta} \quad , \tag{3.17}$$

that implies that $W^M{}_\alpha$ is in the **3875**.

We have thus shown that the algebra can only be consistently deformed if the embedding tensor belongs to $\mathbf{1} \oplus \mathbf{3875}$. In the case of the singlet deformation, W^M_α vanishes and indeed eq. (3.9) becomes the invariance of Θ, which is the Cartan-Killing metric in this case. Therefore our results reproduce the constraints on the embedding tensor found using supersymmetry in [2]. We now show that also the quadratic constraints of [2] follow from the consistency of the deformed E_{11} algebra. These come again from the general analysis of the previous section. In particular, given that Θ is symmetric, both eq. (2.27) for $n = 2$ and eq. (2.28) give the same constraint, which is

$$\Theta^\alpha{}_\beta [f^{\beta\epsilon}{}_\delta \Theta^{\gamma\delta} + f^{\beta\gamma}{}_\delta \Theta^{\delta\epsilon}] = 0 \quad . \tag{3.18}$$

This is the condition that the embedding tensor is invariant when projected by the embedding tensor itself, and corresponds to the condition that the embedding tensor is invariant under the subgroup of E_8 which is gauged.

Here we have considered the Jacobi identities involving the 1-form and 2-form generators, but one can show that also the Jacobi identities involving the 3-forms close if one considers the deformations arising from the embedding tensor in the $\mathbf{1} \oplus \mathbf{3875}$, and more generally the whole E_{11} algebra can be deformed consistently introducing this embedding tensor.

In section 2 we have given a general procedure to compute the field strengths in any dimension. This is expanded in appendix B. In the three-dimensional case, from the group element in eq. (3.7) and the commutators derived in this section one then obtains the field strength for the 1-form,

$$F_{ab,\alpha} = 2[\partial_{[a} A_{b],\alpha} + \frac{1}{2} g \Theta^\beta{}_\delta f^{\delta\gamma}{}_\alpha A_{[a,\beta} A_{b],\gamma} + g W^M{}_\alpha A_{ab,M}] \quad , \tag{3.19}$$

transforming covariantly under the gauge transformations

$$\begin{aligned} \delta A_{a,\alpha} &= \partial_a \Lambda_\alpha - g \Theta^\beta{}_\delta f^{\delta\gamma}{}_\alpha \Lambda_\beta A_{a,\gamma} - g W^M{}_\alpha \Lambda_{a,M} \\ \delta A_{ab,M} &= \partial_{[a} \Lambda_{b],M} + \frac{1}{2} S^{\alpha\beta}_M \partial_{[a} \Lambda_\alpha A_{b],\beta} - g \Theta^\beta{}_\alpha D^\alpha_M{}^N \Lambda_\beta A_{ab,N} \\ &\quad - \frac{g}{2} S^{\alpha\beta}_M W^N{}_\alpha \Lambda_{[a,N} A_{b],\beta} \quad , \end{aligned} \tag{3.20}$$

where $D^\alpha_M{}^N$ are the generators in the **3875**. Given the results in this section, we can also compute the field strength of the 2-forms up to the term involving the 3-form. The result

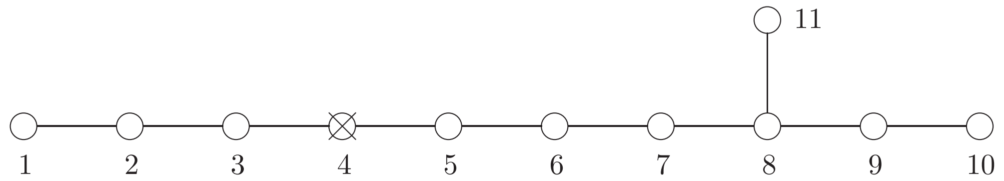

Figure 2. The E_{11} Dynkin diagram corresponding to 4-dimensional supergravity. The internal symmetry group is $E_{7(+7)}$.

is

$$
\begin{aligned}
F_{a_1a_2a_3,M} &= 3\Big[\partial_{[a_1}A_{a_2a_3],M} + \frac{1}{2}S_M^{\alpha\beta}\partial_{[a_1}A_{a_2,\alpha}A_{a_3],\beta} + gA_{[a_1a_2,N}A_{a_3],\alpha}W^N{}_\beta S_M^{\alpha\beta} \\
&\qquad + \frac{g}{6}A_{[a_1,\alpha}A_{a_2,\beta}A_{a_3],\gamma}\Theta^\alpha{}_\delta f^{\delta\beta}{}_\sigma S_M^{\gamma\sigma}\Big] \\
F_{a_1a_2a_3} &= 3\Big[\partial_{[a_1}A_{a_2a_3]} + \frac{1}{2}g^{\alpha\beta}\partial_{[a_1}A_{a_2,\alpha}A_{a_3],\beta} + gA_{[a_1a_2,N}A_{a_3],\alpha}W^N{}_\beta g^{\alpha\beta} \\
&\qquad + \frac{g}{6}A_{[a_1,\alpha}A_{a_2,\beta}A_{a_3],\gamma}\Theta^\alpha{}_\delta f^{\delta\beta\gamma}\Big] \quad .
\end{aligned}
\tag{3.21}
$$

To prove the gauge covariance of these field strengths of the 2-forms one must include the 3-forms and determine their gauge transformations.

To summarise, we have obtained the field strengths and gauge transformations of any gauged maximal supergravity theory in three dimensions. These field strengths satisfy duality conditions. In particular, the field strengths of the 1-forms are related to the derivative of the scalars, while the field strengths of the 2-forms are related to the embedding tensor.

4 D=4

In this section we consider the E_{11} decomposition relevant for the four-dimensional theory. The corresponding Dynkin diagram is shown in figure 2. The global symmetry of four-dimensional massless maximal supergravity [23] is $E_{7(+7)}$. This symmetry rotates electric and magnetic vectors, and as such it is not a symmetry of the lagrangian, but only of the equations of motion. This is in agreement with E_{11}, in which fields and their magnetic duals are treated on the same footing.

The bosonic field content of the supergravity theory contains 70 scalars parametrising the manifold $E_{7(+7)}/\mathrm{SU}(8)$, the metric and 28 vectors, that together with their magnetic duals make the **56** of E_7. E_{11} contains the corresponding generators, together with 2-form generators in the **133** of E_7, whose corresponding fields are dual to the scalars, 3-form generators in the **912**, together with 4-form generators in the **8645** $\oplus$ **133** and an infinite number of generators with mixed symmetry in the spacetime indices. Summarising, the form generators are

$$R^\alpha\ (\mathbf{133}) \quad R^{a,M}\ (\mathbf{56}) \quad R^{a_1a_2,\alpha}\ (\mathbf{133}) \quad R^{a_1a_2a_3,A}\ (\mathbf{912}) \quad R^{a_1\dots a_4,\alpha\beta}\ (\mathbf{8645}\oplus\mathbf{133}) \quad , \tag{4.1}$$

where $\alpha = 1,\dots,133$, $M = 1,\dots,56$ and $A = 1,\dots,912$. The $\alpha\beta$ indices of the 4-form are antisymmetric, which indeed corresponds to the reducible representation $\mathbf{8645}\oplus\mathbf{133}$.

The E_7 algebra is

$$[R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma \quad , \tag{4.2}$$

where $f^{\alpha\beta}{}_\gamma$ are the E_7 structure constants. We also introduce the generators $D^{\alpha}_{M}{}^{N}$ in the $\mathbf{56}$, that satisfy the commutation relation

$$D^{\alpha}_{M}{}^{P} D^{\beta}_{P}{}^{N} - D^{\beta}_{M}{}^{P} D^{\alpha}_{P}{}^{N} = f^{\alpha\beta}{}_\gamma D^{\gamma}_{M}{}^{N} \quad . \tag{4.3}$$

The M indices are raised and lowered by the antisymmetric invariant metric Ω^{MN}, that is for a generic object V^M in the $\mathbf{56}$ we have

$$V^M = \Omega^{MN} V_N \qquad V_M = V^N \Omega_{NM} \quad , \tag{4.4}$$

which implies

$$\Omega^{MN}\Omega_{NP} = -\delta^M_P \quad . \tag{4.5}$$

Raising one index of the generator $D^{\alpha}_{M}{}^{N}$ one gets

$$D^{\alpha,MN} = \Omega^{MP} D^{\alpha}_{P}{}^{N} \quad , \tag{4.6}$$

which is symmetric in MN.

We now write down the rest of the algebra. The commutators between the scalars and the other generators are dictated by the E_7 representation that the generators carry. In particular for the 1-form one has

$$[R^\alpha, R^{a,M}] = D^{\alpha}_{N}{}^{M} R^{a,N} \tag{4.7}$$

and for the 2-form

$$[R^\alpha, R^{a_1a_2,\beta}] = f^{\alpha\beta}{}_\gamma R^{a_1a_2,\gamma} \quad . \tag{4.8}$$

The other commutators are

$$\begin{aligned}
[R^{a_1,M}, R^{A_2,N}] &= D^{MN}_{\alpha} R^{a_1a_2,\alpha} \\
[R^{a_1,M}, R^{a_2a_3,\alpha}] &= S^{M\alpha}_{A} R^{a_1a_2a_3,A} \\
[R^{a_1a_2,\alpha}, R^{a_3a_4,\beta}] &= R^{a_1\dots a_4,\alpha\beta} \\
[R^{a_1,M}, R^{a_2a_3a_4,A}] &= C^{MA}_{\alpha\beta} R^{a_1\dots a_4,\alpha\beta} \quad ,
\end{aligned} \tag{4.9}$$

where we have introduced the two E_7 invariant tensors $S^{M\alpha}_{A}$ and $C^{MA}_{\alpha\beta}$, the last one being antisymmetric in $\alpha\beta$. Following [3], we are using the metric

$$g^{\alpha\beta} = D^{\alpha}_{M}{}^{N} D^{\beta}_{N}{}^{M} \tag{4.10}$$

to raise and lower indices in the adjoint. This metric is proportional to the Cartan-Killing metric, as can be seen from

$$f_{\alpha\beta\gamma} f^{\alpha\beta\delta} = -3\delta_\gamma^\delta \quad . \tag{4.11}$$

A summary of the conventions for E_7 and E_6 is given in appendix A. The Jacobi identity involving three 1-forms produces the condition

$$D_\alpha^{(MN} S_A^{P)\alpha} = 0 \quad , \tag{4.12}$$

and $S_A^{M\alpha}$ also satisfies

$$D_{\alpha,M}{}^N S_A^{M\alpha} = 0 \quad , \tag{4.13}$$

which can be deduced from the fact that there is no singlet in the tensor product $\mathbf{56} \otimes \mathbf{912}$. Contracting eq. (4.12) with D_{NP}^β gives

$$S_A^{M\alpha} + 2(D^\alpha D_\beta)_N{}^M S_A^{N\beta} = 0 \quad . \tag{4.14}$$

As will be described in detail in appendix A, the conditions of eqs. (4.13) and (4.14) project the $M\alpha$ indices of $S_A^{M\alpha}$ along the $\mathbf{912}$. Indeed, the only way of building an invariant from tensoring a $\mathbf{912}$ index with the product $\mathbf{56} \otimes \mathbf{133}$ is that this product is projected on the $\mathbf{912}$. The Jacobi identity between $R^{a,M}$, $R^{b,N}$ and $R^{cd,\alpha}$ gives the condition

$$S_A^{M\alpha} C_{\beta\gamma}^{NA} + S_A^{N\alpha} C_{\beta\gamma}^{MA} + \delta_{[\beta}^\alpha D_{\gamma]}^{MN} = 0 \quad . \tag{4.15}$$

One can check that also all the other Jacobi identities are satisfied. We also define the invariant tensor Ω_{AB} in the antisymmetric product of two $\mathbf{912}$ representations, using the relation

$$S_A^{M\alpha} S_{M\alpha B} = \Omega_{AB} \quad , \tag{4.16}$$

and we use Ω_{AB} to raise and lower indices in the $\mathbf{912}$, adopting conventions analogous to those of eq. (4.4).

Writing down the group element

$$g = e^{x\cdot P} e^{A_{a_1\ldots a_4,\alpha\beta} R^{a_1\ldots a_4,\alpha\beta}} \ldots e^{A_{a,M} R^{a,M}} e^{\phi_\alpha R^\alpha} \tag{4.17}$$

one determines the field strengths of the massless theory by antisymmetrising the spacetime indices of the various terms in the Maurer-Cartan form, and the field equations of the supergravity theory arise as duality relations. In particular, the field-strength of the vector satisfies self-duality conditions, while the field-strength of the 2-form in dual to the scalar derivative. The field-strengths of the 3-forms vanish in the massless theory. In deriving the field strengths of the massless theory one takes the positive level E_{11} generators to commute with momentum. In the following we will consider the deformation of the E_{11} algebra which results from modifying the commutation relations of the E_{11} generators with momentum compatibly with the Jacobi identities, following the general results of section 2.

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Applying the general analysis of section 2 to the four-dimensional case, one makes the identifications

$$
\begin{aligned}
R^{a_1,M_1} &\to R^{a_1,M} & \Theta^{M_1}{}_{\alpha} &\to \Theta^{M}_{\alpha} \\
R^{a_1a_2,M_2} &\to R^{a_1a_2,\alpha} & W^{M_2}{}_{M_1} &\to W^{\alpha}_{(2)\,M} \\
R^{a_1a_2a_3,M_3} &\to R^{a_1a_2a_3,A} & W^{M_3}{}_{M_2} &\to W^{A}_{(3)\,\alpha} \\
R^{a_1\dots a_4,M_4} &\to R^{a_1\dots a_4,\alpha\beta} & W^{M_4}{}_{M_3} &\to W^{\alpha\beta}_{(4)\ A} \quad .
\end{aligned}
\tag{4.18}
$$

Eq. (2.19), arising from the Jacobi identity between two 1-forms and momentum, reads in this case

$$W^{\alpha}_{(2)\,P} D^{MN}_{\alpha} = 2X^{(MN)}{}_{P} \quad , \tag{4.19}$$

where

$$X^{MN}{}_{P} = \Theta^{M}_{\alpha} D^{\alpha N}_{P} \quad . \tag{4.20}$$

Using eq. (4.10), from eq. (4.19) one gets

$$W^{\alpha}_{(2)\,M} = -2D^{\beta\ N}_{M} D^{\alpha}_{NP} \Theta^{P}_{\beta} \quad . \tag{4.21}$$

Eq. (2.16) for $n = 2$, which is the condition that the Jacobi identity involving the 1-form, the 2-form and momentum is satisfied, reads

$$S^{M\alpha}_{A} W^{A}_{(3)\,\beta} = -W^{\alpha}_{(2)\,N} D^{MN}_{\beta} + \Theta^{M}_{\gamma} f^{\gamma\alpha}{}_{\beta} \quad . \tag{4.22}$$

This has to be compatible with the conditions of eqs. (4.13) and (4.12) that S satisfies. The first condition gives

$$2\Theta^{M}_{\alpha} D_{\beta,M}{}^{N} D^{\alpha}_{N}{}^{P} + \Theta^{P}_{\beta} - X^{MN}{}_{M} D_{\beta,N}{}^{P} = 0 \quad , \tag{4.23}$$

while the second is identically satisfied. If we then contract this last equation with $D^{\beta Q}_{P}$ we get

$$X^{MN}{}_{M} = \Theta^{M}_{\alpha} D^{\alpha}_{M}{}^{N} = 0 \quad , \tag{4.24}$$

and plugging this into (4.23) and comparing with (4.21) one obtains

$$W^{\alpha}_{(2)\,M} = -\Theta^{\alpha}_{M} \quad . \tag{4.25}$$

Substituting this in eq. (4.19) gives

$$X^{(MNP)} = 0 \quad , \tag{4.26}$$

and contracting this with $D_{\alpha,NP}$ gives

$$\Theta^{M}_{\alpha} = -2D_{\alpha,N}{}^{P} \Theta^{N}_{\beta} D^{\beta M}_{P} \quad . \tag{4.27}$$

The two conditions of eqs. (4.24) and (4.27) project the embedding tensor Θ to belong to the **912** of E_7. Therefore we have shown that E_{11} produces all the linear (or representation) constraints on Θ^{M}_{α}. The Jacobi identities at the next level then give

$$W^{\alpha\beta}_{(4)\ A} = -2\Theta^{[\alpha}_{M} S^{M\beta]}_{A} \quad . \tag{4.28}$$

The embedding tensor also satisfies quadratic constraints that follow from the general analysis of section 2. In particular eq. (2.28), resulting from the Jacobi identity involving $\Theta_\alpha^M R^\alpha$, $R^{a,M}$ and momentum, together with eq. (2.27) for $n = 2$, resulting from the Jacobi identity involving the 2-form and two momentum operators, and which reads in this case

$$\Omega_{MN}\Theta_\alpha^M\Theta_\beta^N = 0 \quad , \tag{4.29}$$

imply that the quantities $(X^M)^N{}_P$ are the generators of the subgroup of E_7 that in gauged. This analysis thus exactly reproduces all the constraints of [7]. It is important to stress that from E_{11} all the constraints arise from imposing the consistency of the deformed algebra.

To summarise, the Jacobi identities impose that the commutators of the deformed E_{11} generators with momentum are

$$\begin{aligned}
[R^{a,M}, P_b] &= -g\delta_b^a\Theta_\alpha^M R^\alpha \\
[R^{a_1a_2,\alpha}, P_b] &= g\Theta_M^\alpha \delta_b^{[a_1} R^{a_2],M} \\
[R^{a_1a_2a_3,A}, P_b] &= -gS^A_{M\alpha}[\Theta_N^\alpha D_\beta^{MN} + \Theta_\gamma^M f^{\gamma\alpha}{}_\beta]\delta_b^{[a_1} R^{a_2a_3],\beta} \\
[R^{a_1\dots a_4,\alpha\beta}, P_b] &= 2g\Theta_M^{[\alpha} S_A^{M\beta]}\delta_b^{[a_1} R^{a_2\dots a_4],A} \quad .
\end{aligned} \tag{4.30}$$

From these commutators, as well as the E_{11} commutators of eq. (4.9), and using the group element in eq. (4.17), one determines the field strengths and gauge transformations of the fields. The result is

$$\begin{aligned}
F_{a_1a_2,M} &= 2\Big[\partial_{[a_1}A_{a_2],M} - g\Theta_M^\alpha A_{a_1a_2,\alpha} + \frac{g}{2}A_{[a_1,N}A_{a_2],P}X^{NP}{}_M\Big] \\
F_{a_1a_2a_3,\alpha} &= 3\Big[\partial_{[a_1}A_{a_2a_3],\alpha} + \frac{1}{2}\partial_{[a_1}A_{a_2,M}A_{a_3],N}D_\alpha^{MN} \\
&\quad + gS^A_{M\beta}(\Theta_N^\beta D_\alpha^{MN} + \Theta_\gamma^M f^{\gamma\beta}{}_\alpha)A_{a_1a_2a_3,A} \\
&\quad - g\Theta_M^\beta D_\alpha^{MN}A_{[a_1a_2,\beta}A_{a_3],N} + \frac{g}{6}A_{[a_1,M}A_{a_2,N}A_{a_3],P}X^{MN}{}_Q D_\alpha^{QP}\Big] \\
F_{a_1\dots a_4,A} &= 4\Big[\partial_{[a_1}A_{a_2a_3a_4],A} - S_A^{M\alpha}\partial_{[a_1}A_{a_2a_3,\alpha}A_{a_4],M} - \frac{1}{6}D_\alpha^{MN}S_A^{P\alpha}\partial_{[a_1}A_{a_2,M}A_{a_3,N}A_{a_4],P} \\
&\quad - 2g\Theta_M^\alpha S_A^{M\beta}A_{a_1\dots a_4,\alpha\beta} - gS^B_{M\alpha}(\Theta_N^\alpha D_\beta^{MN} + \Theta_\gamma^M f^{\gamma\alpha}{}_\beta)S_A^{P\beta}A_{[a_1a_2a_3,B}A_{a_4],P} \\
&\quad - \frac{g}{2}\Theta_M^\alpha S_A^{M\alpha}A_{[a_1a_2,\alpha}A_{a_3a_4],\beta} + \frac{g}{2}\Theta_P^\alpha D_\beta^{MP}S_A^{N\beta}A_{[a_1a_2,\alpha}A_{a_3,M}a_{a_4],N} \\
&\quad - \frac{g}{24}X^{MN}{}_R D_\alpha^{PR}S_A^{Q\alpha}A_{[a_1,M}A_{a_2,N}A_{a_3,P}A_{a_4],Q}\Big] \quad ,
\end{aligned} \tag{4.31}$$

transforming covariantly under

$$\begin{aligned}
\delta A_{a,M} &= a_{a,M} - gX^{NP}{}_M\Lambda_N A_{a,P} \\
\delta A_{a_1a_2,\alpha} &= a_{a_1a_2,\alpha} + \frac{1}{2}D_\alpha^{MN}a_{[a_1,M}A_{a_2],N} - g\Theta_\beta^M\Lambda_M f^{\beta\gamma}{}_\alpha A_{a_1a_2,\gamma} \\
\delta A_{a_1a_2a_3,A} &= a_{a_1a_2a_3,A} + S_A^{M\alpha}a_{[a_1,M}A_{a_2a_3],\alpha} - \frac{1}{6}S_A^{M\alpha}D_\alpha^{NP}A_{[a_1,M}A_{a_2,N}a_{a_3],P} \\
&\quad - g\Theta_\alpha^M D_A^{\alpha B}\Lambda_M A_{a_1a_2a_3,B}
\end{aligned} \tag{4.32}$$

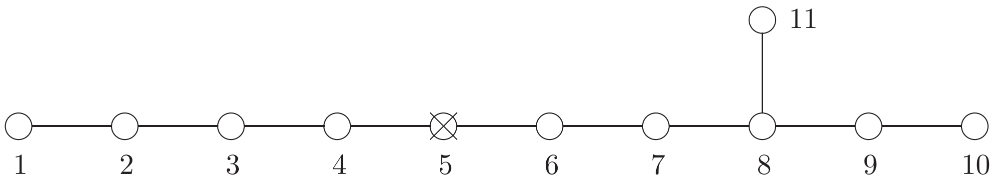

Figure 3. The E_{11} Dynkin diagram corresponding to 5-dimensional supergravity. The internal symmetry group is $E_{6(+6)}$.

$$
\begin{aligned}
\delta A_{a_1\ldots a_4,\alpha\beta} = &\, \partial_{[a_1}\Lambda_{a_2a_3a_4],\alpha\beta} + \frac{1}{2} a_{[a_1a_2,\alpha}A_{a_3a_4],\beta} + C^{MA}_{\alpha\beta} a_{[a_1,M}A_{a_2a_3a_4],A} \\
& - \frac{1}{24} D^{PQ}_{\gamma} C^{MA}_{\alpha\beta} S^{N\gamma}_{A} A_{[a_1,M}A_{a_2,N}A_{a_3,P}a_{a_4],Q} + \frac{1}{4} D^{MN}_{[\alpha} A_{[a_1a_2,\beta]}A_{a_3,M}a_{a_4],N} \\
& + 2g\Theta^M_\delta f^{\delta\gamma}{}_{[\alpha}\Lambda_M A_{a_1\ldots a_4,\beta]\gamma} \quad , \qquad (4.33)
\end{aligned}
$$

where $D^{\alpha B}_{A}$ are the generators in the **912** and the parameters $a_{a,M}$, $a_{a_1a_2,\alpha}$ and $a_{a_1a_2a_3,A}$ are defined in terms of the gauge parameters as

$$
\begin{aligned}
a_{a,M} &= \partial_a\Lambda_M + g\Theta^\alpha_M\Lambda_{a,\alpha} \\
a_{a_1a_2,\alpha} &= \partial_{[a_1}\Lambda_{a_2],\alpha} - gS^A_{M\beta}(\Theta^\beta_N D^{MN}_\alpha + \Theta^M_\gamma f^{\gamma\beta}{}_\alpha)\Lambda_{a_1a_2,A} \\
a_{a_1a_2a_3,A} &= \partial_{[a_1}\Lambda_{a_2a_3],A} + 2g\Theta^\alpha_M S^{M\beta}_A \Lambda_{a_1a_2a_3,\alpha\beta} \quad . \qquad (4.34)
\end{aligned}
$$

These are the field strengths and gauge transformations of any gauged maximal supergravity theory in four dimensions.

5 D=5

We now consider the five-dimensional case. The bosonic sector of the maximal massless supergravity theory in five dimensions [24] contains 42 scalars parametrising the manifold $E_{6(+6)}/USp(8)$, the metric and a 1-form in the **27**. This theory arises from the decomposition of E_{11} appropriate to five dimensions whose Dynkin diagram is shown in figure 3. The form generators up to the 4-form included that occur in this decomposition of E_{11} with respect to $GL(5,\mathbb{R}) \otimes E_6$ are [13]

$$
R^\alpha\ (\mathbf{78}) \quad R^{a,M}\ (\overline{\mathbf{27}}) \quad R^{ab}{}_M\ (\mathbf{27}) \quad R^{abc,\alpha}\ (\mathbf{78}) \quad R^{abcd}{}_{MN}\ (\overline{\mathbf{351}}) \quad , \qquad (5.1)
$$

where R^α, $\alpha = 1, \ldots, 78$ are the E_6 generators, and an upstairs M index, $M = 1, \ldots, 27$, corresponds to the $\overline{\mathbf{27}}$ representation of E_6, a downstairs M index to the **27** of E_6 and the 4-forms are antisymmetric in MN, thus belonging to the $\overline{\mathbf{351}}$. The commutation relations for the E_6 generators is

$$
[R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma \quad , \qquad (5.2)
$$

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where $f^{\alpha\beta}{}_\gamma$ are the structure constants of E_6. The commutation relations of R^α with all the other generators is determined by the E_6 representations that they carry. This gives

$$\begin{aligned}
[R^\alpha, R^{a,M}] &= (D^\alpha)_N{}^M R^{a,N} \\
[R^\alpha, R^{ab}{}_M] &= -(D^\alpha)_M{}^N R^{ab}{}_N \\
[R^\alpha, R^{abc,\beta}] &= f^{\alpha\beta}{}_\gamma R^{abc,\gamma} \\
[R^\alpha, R^{abcd}{}_{MN}] &= -(D^\alpha)_M{}^P R^{abcd}{}_{PN} - (D^\alpha)_N{}^P R^{abcd}{}_{MP} \quad ,
\end{aligned} \tag{5.3}$$

where $(D^\alpha)_N{}^M$ obey

$$[D^\alpha, D^\beta]_M{}^N = f^{\alpha\beta}{}_\gamma (D^\gamma)_M{}^N \quad . \tag{5.4}$$

The commutation relations of all the other generators are

$$\begin{aligned}
[R^{a,M}, R^{b,N}] &= d^{MNP} R^{ab}{}_P \\
[R^{a,N}, R^{bc}{}_M] &= g_{\alpha\beta}(D^\alpha)_M{}^N R^{abc,\beta} \\
[R^{ab}{}_M, R^{cd}{}_N] &= R^{abcd}{}_{MN} \\
[R^{a,P}, R^{bcd,\alpha}] &= S^{\alpha P,MN} R^{abcd}{}_{MN}
\end{aligned} \tag{5.5}$$

where d^{MNP} is the completely symmetric invariant tensor of E_6 and $g^{\alpha\beta}$ is defined by the relation

$$D^\alpha_M{}^N D^\beta_N{}^M = g^{\alpha\beta} \tag{5.6}$$

and is thus proportional to the Cartan-Killing metric of E_6, and is the metric that is used to raise and lower indices in the adjoint (we are using the E_6 conventions of [3], that are summarised in appendix A). Another useful identity is

$$f_{\alpha\beta\gamma} f^{\alpha\beta\delta} = -4\delta^\delta_\gamma \quad , \tag{5.7}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of E_6. $S^{\alpha P,MN}$ is also an invariant tensor, antisymmetric with respect to MN, and the Jacobi identity between two 1-forms and one 2-form gives

$$g_{\alpha\beta} D^{\alpha}_Q{}^{(P} S^{\beta R),MN} = -\frac{1}{2}\delta^{[M}_Q d^{N]PR} \quad . \tag{5.8}$$

Using the fact that d^{MNP} is completely antisymmetric one derives from this the condition

$$g_{\alpha\beta} D^\alpha_M{}^N S^{\beta M,PQ} = 0 \quad . \tag{5.9}$$

One can show that all the Jacobi identities involving the generators are satisfied using the commutators listed above [13].

We now show that $S^{\alpha M,NP}$ is proportional to $D^\alpha_Q{}^{[N} d^{P]MQ}$ and determine the coefficient of proportionality. We introduce the invariant tensor d_{MNP} in the completely symmetric product of three **27** indices, that satisfies [4]

$$d^{MNP} d_{MNQ} = \delta^P_Q \quad . \tag{5.10}$$

Observe that the normalisation used here differs from the one used in [16], where the same contraction produced the delta function with a factor 5. This simply corresponds to a rescaling of d by $\sqrt{5}$. In appendix A we derive the useful relation

$$g_{\alpha\beta} D^{\alpha}_{M}{}^{N} D^{\beta}_{P}{}^{Q} = \frac{1}{6}\delta^{N}_{P}\delta^{Q}_{M} + \frac{1}{18}\delta^{N}_{M}\delta^{Q}_{P} - \frac{5}{3}d^{NQR}d_{MPR} \quad . \tag{5.11}$$

Using this relation one shows that eq. (5.8) implies

$$S^{\alpha M,NP} = -3D^{\alpha}_{Q}{}^{[N} d^{P]MQ} \quad . \tag{5.12}$$

Notice that this relation differs from the one in [16] because of the different conventions used in that paper. In particular in [16] the generators were normalised in such a way that the first coefficient in eq. (5.11) was equal to 1. Contracting eq. (5.8) with $D^{\gamma}_{P}{}^{Q}$ and using eq. (5.12) one finally gets

$$S^{\alpha M,NP} + \frac{3}{2}(D^{\alpha}D_{\beta})_{Q}{}^{M} S^{\beta Q,NP} = 0 \quad . \tag{5.13}$$

As we will describe in detail in appendix A, the conditions of eqs. (5.9) and (5.13) are the conditions that the indices αM in $S^{\alpha M,NP}$ are in the $\overline{\mathbf{351}}$. Indeed, given that the NP indices are antisymmetric and thus form the $\mathbf{351}$, the only way of building an invariant tensor from tensoring this with the product $\overline{\mathbf{27}} \otimes \mathbf{78}$ is to project this product on the $\overline{\mathbf{351}}$. Later in this section we will derive from E_{11} the same projection rules for the embedding tensor.

From the group element

$$g = e^{x\cdot P} e^{A^{MN}_{a_1\dots a_4} R^{a_1\dots a_4}_{MN}} e^{A_{a_1 a_2 a_3,\alpha} R^{a_1 a_2 a_3,\alpha}} e^{A^{M}_{a_1 a_2} R^{a_1 a_2}_{M}} e^{A_{a,M} R^{a,M}} e^{\phi_\alpha R^\alpha} \quad , \tag{5.14}$$

one can compute the Maurer-Cartan form using the fact that the generators commute with momentum in the massless theory. The complete antisymmetrisation of the indices leads to the gauge-invariant field-strengths of the massless theory obtained in [13].

We now consider the deformed case. This was analysed in detail in [17], where it was shown that introducing the Ogievetsky generators and deforming the algebra one obtains the field strengths of all the fields and dual fields of the gauged maximal five-dimensional supergravity which had been previously obtained in [16]. We now only concentrate on the deformed E_{11} generators, as we do in all other cases in this paper, which is all one needs to determine the field strengths of the massive theory. This is completely consistent if one simply assumes that the indices are antisymmetrised, and indeed the completely antisymmetric part of the Ogievetsky generators vanishes. As it is clear from the discussion in section 2, considering only the constraints coming from taking into account the deformed E_{11} generators is enough to determine the whole deformed algebra. The following analysis thus determines all the possible massive deformations of the algebra of eq. (5.5).

The general analysis of section 2 can be applied to the five-dimensional case making the identifications

$$\begin{aligned}
R^{a_1,M_1} &\to R^{a_1,M} & \Theta^{M_1}{}_\alpha &\to \Theta^M_\alpha \\
R^{a_1a_2,M_2} &\to R^{a_1a_2}{}_M & W^{M_2}{}_{N_1} &\to W_{MN} \\
R^{a_1a_2a_3,M_3} &\to R^{a_1a_2a_3,\alpha} & W^{M_3}{}_{M_2} &\to W^{\alpha M}_{(3)} \\
R^{a_1\dots a_4,M_4} &\to R^{a_1\dots a_4}{}_{MN} & W^{M_4}{}_{M_3} &\to W_{(4)MN\alpha} \quad .
\end{aligned} \tag{5.15}$$

The Jacobi identity among two 1-forms and momentum gives eq. (2.19), which in this case is

$$d^{MNQ}W_{QP} = 2X^{(MN)}{}_P \quad , \tag{5.16}$$

where as usual

$$X^{MN}{}_P = \Theta^M_\alpha D^{\alpha N}_P \quad . \tag{5.17}$$

Contracting with d_{MNR} one then gets

$$W_{RP} = 2d_{MN[R}\Theta^M_\alpha D^{\alpha}_{P]}{}^N - d_{RPN}X^{MN}{}_M \quad , \tag{5.18}$$

where the first term is antisymmetric and the second is symmetric in RP. The Jacobi identity involving the 1-form, the 2-form and momentum gives eq. (2.16) for $n = 2$, which is

$$W_{NP}d^{PMQ} + X^{MQ}{}_N = -W^Q_{(3)\alpha}D^{\alpha}_N{}^M \quad , \tag{5.19}$$

while the Jacobi identity involving two 2-forms and momentum gives

$$W^\alpha_{(4)MN} = W_{MP}D^\alpha_N{}^P - W_{NP}D^\alpha_M{}^P \tag{5.20}$$

and the Jacobi identity involving the 1-form, the 3-form and momentum gives

$$W^\gamma_{(4)NP}S^{M,NP}_\alpha = \Theta^M_\beta f^\beta{}_{\alpha\gamma} - W^N_{(3)\alpha}D_{\gamma,N}{}^M \quad . \tag{5.21}$$

Substituting $W^\alpha_{(4)MN}$ given in eq. (5.20) in this last equation and contracting α and γ gives

$$2W_{NQ}D_{\alpha,P}{}^QS^{\alpha M,NP} = -W^N_{(3)\alpha}D_{\alpha,N}{}^M \quad , \tag{5.22}$$

and using eq. (5.19), as well as eqs. (5.12) and (5.11), one obtains

$$W_{MN}d^{MNP} = 0 \quad . \tag{5.23}$$

From eq. (5.19) one can deduce that this implies

$$X^{MN}{}_M = D^\alpha_M{}^N\Theta^M_\alpha = 0 \quad , \tag{5.24}$$

so that from eq. (5.18) one gets that W_{MN} must be antisymmetric, that is it belongs to the $\overline{\mathbf{351}}$. This also implies that

$$W^M_{(3)\alpha} = \Theta^M_\alpha \quad , \tag{5.25}$$

and substituting everything in eq. (5.21) one gets

$$f^{\alpha\beta}{}_{\gamma}\Theta^{Q}_{\beta} - D^{\alpha}_{P}{}^{Q}\Theta^{P}_{\gamma} = 2D^{\alpha}_{M}{}^{P}W_{PN}g_{\beta\gamma}S^{\beta Q,MN} \quad , \tag{5.26}$$

where W_{MN} is in the $\overline{\mathbf{351}}$. If Θ was not along the $\overline{\mathbf{351}}$ this equation would be inconsistent because it would imply the invariance of Θ under E_6. Therefore the embedding tensor has to belong to the $\overline{\mathbf{351}}$. To determine this more rigorously, we now show that eq. (5.26) leads to the projection for Θ analogous to that in eq. (5.13). Contracting eq. (5.26) with $D^{\alpha}_{Q}{}^{R}$ gives

$$\frac{26}{9}\Theta^{R}_{\gamma} + (D_{\gamma}D^{\beta})_{Q}{}^{R}\Theta^{Q}_{\beta} = \frac{20}{9}W_{NP}S^{R,NP}_{\gamma} \quad , \tag{5.27}$$

while contracting it with $f_{\alpha\gamma\delta}$ gives

$$4\Theta^{R}_{\gamma} + (D_{\gamma}D^{\beta})_{Q}{}^{R}\Theta^{Q}_{\beta} = \frac{10}{3}W_{NP}S^{R,NP}_{\gamma} \quad , \tag{5.28}$$

and combining these two equations one gets

$$\Theta^{R}_{\gamma} + \frac{3}{2}(D_{\gamma}D^{\beta})_{Q}{}^{R}\Theta^{Q}_{\beta} = 0 \quad . \tag{5.29}$$

This equation, together with eq. (5.24), projects the embedding tensor on the $\overline{\mathbf{351}}$.

The embedding tensor also satisfies quadratic constraints, as discussed in complete generality in section 2. The Jacobi identity between $\Theta^{M}_{\alpha}R^{\alpha}$, $R^{a,N}$ and P_b gives

$$\Theta^{M}_{\alpha}\Theta^{N}_{\beta}f^{\alpha\beta}{}_{\gamma} - \Theta^{P}_{\gamma}X^{MN}{}_{P} = 0 \quad , \tag{5.30}$$

while the Jacobi identity between the 2-form and two momentum operators gives

$$\Theta^{M}_{\alpha}W_{MN} = 0 \quad . \tag{5.31}$$

Combining these two conditions one obtains the condition that the embedding tensor is invariant under the gauge group, which is the subgroup of E_6 generated by $\Theta^{M}_{\alpha}R^{\alpha}$.

To summarise, we have shown that the Jacobi identities constrain the commutators of the deformed E_{11} generators and momentum to be

$$\begin{aligned}
[R^{a,M}, P_b] &= -g\delta^{a}_{b}\Theta^{M}_{\alpha}R^{\alpha} \\
[R^{a_1a_2}{}_{M}, P_b] &= -gW_{MN}\delta^{[a_1}_{b}R^{a_2],N} \\
[R^{a_1a_2a_3}{}_{\alpha}, P_b] &= -g\Theta^{M}_{\alpha}\delta^{[a_1}_{b}R^{a_2a_3]}{}_{M} \\
[R^{a_1\dots a_4}{}_{MN}, P_b] &= -2gW_{[M|P|}D^{\alpha}_{N]}{}^{P}\delta^{[a_1}_{b}R^{a_2a_3a_4]}{}_{\alpha} \quad .
\end{aligned} \tag{5.32}$$

From this commutators,as well as the E_{11} commutators of eq. (5.5), and using the group element of eq. (5.14), we determine the field strengths of the fields using the general results of section 2. The result is

$$
\begin{aligned}
F_{a_1a_2,M} &= 2\Big[\partial_{[a_1}A_{a_2],M} + \frac{1}{2}gX_M^{[NP]}A_{[a_1,N}A_{a_2],P} - gW_{MN}A^N_{a_1a_2}\Big]\\
F^M_{a_1a_2a_3} &= 3\Big[\partial_{[a_1}A^M_{a_2a_3]} + \frac{1}{2}\partial_{[a_1}A_{a_2,N}A_{a_3],P}d^{MNP} - 2gX_P^{(MN)}A^P_{[a_1a_2}A_{a_3],N}\\
&\qquad + \frac{1}{6}gX_R^{[NP]}d^{RQM}A_{[a_1,N}A_{a_2,P}A_{a_3],Q} + g\Theta^M_\alpha A^\alpha_{a_1a_2a_3}\Big]\\
F^\alpha_{a_1\dots a_4} &= 4\Big[\partial_{[a_1}A^\alpha_{a_2\dots a_4]} - \frac{1}{6}\partial_{[a_1}A_{a_2,M}A_{a_3,N}A_{a_4],P}d^{MNQ}D_Q^{\alpha\ P} - \partial_{[a_1}A^M_{a_2a_3}A_{a_4],N}D_M^{\alpha\ N}\\
&\qquad + gD_M^{\alpha\ P}\Theta^M_\beta A_{[a_1,P}A^\beta_{a_2\dots a_4]} + 2gD_M^{\alpha\ P}W_{PN}A^{MN}_{a_1\dots a_4}\\
&\qquad - \frac{g}{2}D_M^{\alpha\ P}W_{PN}A^M_{[a_1a_2}A^N_{a_3a_4]} - gD_M^{\alpha\ P}X_Q^{(MR)}A_{[a_1,P}A_{a_2,R}A^Q_{a_3a_4]}\\
&\qquad - \frac{1}{24}gX_R^{[MN]}d^{RPS}D_S^{\alpha Q}A_{[a_1,M}A_{a_2,N}A_{a_3,P}A_{a_4],Q}\Big] \quad . \qquad (5.33)
\end{aligned}
$$

These are the field-strengths of the five-dimensional gauged maximal supergravity [16]. One can also derive the gauge transformations of the fields from the non-linear realisation. The result is

$$
\begin{aligned}
\delta A_{a,M} &= a_{a,M} - g\Lambda_N X^{NP}{}_M A_{a,P}\\
\delta A^M_{a_1a_2} &= a^M_{a_1a_2} + \frac{1}{2}a_{[a_1,N}A_{a_2]P}d^{MNP} + g\Lambda_N X^{NM}{}_P A^P_{a_1a_2}\\
\delta A^\alpha_{a_1a_2a_3} &= a^\alpha_{a_1a_2a_3} + a_{[a_1,M}A^N_{a_2a_3]}D_N^{\alpha\ M} + \frac{1}{6}a_{[a_1,M}A_{a_2,N}A_{a_3],P}d^{MNQ}D_Q^{\alpha\ P}\\
&\qquad - g\Lambda_M\Theta^M_\beta f^{\alpha\beta}{}_\gamma A^\gamma_{a_1a_2a_3}\\
\delta A^{MN}_{a_1\dots a_4} &= \partial_{[a_1}\Lambda^{MN}_{a_2a_3a_4]} + \frac{1}{2}a^{[M}_{[a_1a_2}A^{N]}_{a_3a_4} + a_{[a_1,P}A^\alpha_{a_2a_3a_4]}g_{\alpha\beta}S^{\beta P,MN}\\
&\qquad - \frac{1}{24}a_{[a_1,P}A_{a_2,Q}A_{a_3,R}A_{a_4],S}d^{PQT}D_T^{\alpha\ R}S^{\beta S,MN}g_{\alpha\beta} - \frac{1}{4}a_{[a_1,P}A_{a_2,Q}A^{[M}_{a_3a_4]}d^{N]PQ}\\
&\qquad - 2g\Lambda_P X^{P[M}{}_Q A^{N]Q}_{a_1\dots a_4} \quad , \qquad (5.34)
\end{aligned}
$$

where the parameters $a_{a,M}$, $a^M_{a_1a_2}$ and $a^\alpha_{a_1a_2a_3}$ are defined in terms of the gauge parameters as

$$
\begin{aligned}
a_{a,M} &= \partial_a\Lambda_M + gW_{MN}\Lambda^N_a\\
a^M_{a_1a_2} &= \partial_{[a_1}\Lambda^M_{a_2]} - g\Theta^M_\alpha\Lambda^\alpha_{a_1a_2}\\
a^\alpha_{a_1a_2a_3} &= \partial_{[a_1}\Lambda^\alpha_{a_2a_3]} - 2gW_{MP}D_N^{\alpha\ P}\Lambda^{MN}_{a_1a_2a_3} \quad . \qquad (5.35)
\end{aligned}
$$

We also compute the field strength of the 4-form up to a term involving the 5-form, the

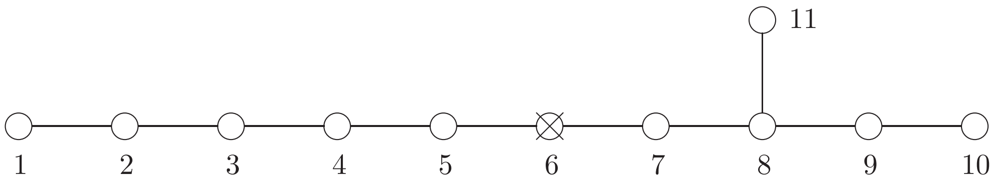

Figure 4. The E_{11} Dynkin diagram corresponding to 6-dimensional supergravity. The internal symmetry group is SO(5, 5).

result being

$$
\begin{aligned}
F^{MN}_{a_1\dots a_5} = 5\Big[&\partial_{[a_1} A^{MN}_{a_2\dots a_5]} + \partial_{[a_1} A^{\alpha}_{a_2\dots a_4} A_{a_5],P} S^{\beta P,MN} g_{\alpha\beta} \\
&-\frac{1}{2} D_{\alpha P}{}^{Q} S^{\alpha R,MN} \partial_{[a_1} A^{P}_{a_2 a_3} A_{a_4,Q} A_{a_5],R} + \frac{1}{2} \partial_{[a_1} A^{[M}_{a_2 a_3} A^{N]}_{a_4 a_5]} \\
&-\frac{1}{24} d^{PRQ} D_{\alpha R}{}^{S} S^{\alpha T,MN} \partial_{[a_1} A_{a_2,P} A_{a_3,Q} A_{a_4,S} A_{a_5],T} \\
&+2g W_{RQ} D_{\alpha S}{}^{Q} S^{\alpha P,MN} A^{RS}_{[a_1\dots a_4} A_{a_5],P} + g \Theta^{[M}_{\alpha} A^{\alpha}_{[a_1 a_2 a_3} A^{N]}_{a_4 a_5]} \\
&-\frac{g}{2} \Theta^{P}_{\alpha} D_{\beta,P}{}^{Q} S^{\beta R,MN} A^{\alpha}_{[a_1 a_2 a_3} A_{a_4,Q} A_{a_5],R} \\
&+\frac{g}{2} W_{RQ} D_{\alpha,S}{}^{Q} S^{\alpha P,MN} A^{R}_{[a_1 a_2} A^{S}_{a_3 a_4} A_{a_5],P} \\
&-\frac{g}{6} W_{TR} d^{RUS} D_{\alpha S}{}^{P} S^{\alpha Q,MN} A^{T}_{[a_1 a_2} A_{a_3,U} A_{a_4,P} A_{a_5],Q} \\
&-\frac{g}{5!} X^{PQ}{}_{U} d^{URV} D_{\beta,V}{}^{S} S^{\beta T,MN} A_{[a_1,P} A_{a_2,Q} A_{a_3,R} A_{a_4,S} A_{a_5],T}\Big] \quad . \qquad (5.36)
\end{aligned}
$$

In order to compute the complete field strength for the 4-form one should consider the contribution of the 5-form generators in the deformed algebra.

6 D=6

In this section we consider the six-dimensional case. The symmetry of the massless maximal supergravity theory in 6 dimensions [25] is SO(5, 5), and the bosonic sector of the theory describes 25 scalars parametrising SO(5, 5)/[SO(5) × SO(5)], the metric, a 1-form in the **16** and a 2-form in the **10**, whose field strength satisfies a self-duality condition. From E_{11} this theory arises after deleting node 6 in the E_{11} Dynkin diagram, as shown in figure 4. From the diagram it is manifest that the 1-forms belong to the spinor representation.

The positive level E_{11} generators with completely antisymmetric spacetime indices that arise in six dimensions, without considering the 6-forms, are

$$
\begin{gathered}
R^{MN} \quad (\mathbf{45}) \qquad R^{a,\dot\alpha} \quad (\overline{\mathbf{16}}) \qquad R^{a_1 a_2,M} \quad (\mathbf{10}) \qquad R^{a_1 a_2 a_3,\alpha} \quad (\mathbf{16}) \\
R^{a_1 a_2 a_3 a_4,MN} \quad (\mathbf{45}) \qquad R^{a_1\dots a_5,M\alpha} \quad (\overline{\mathbf{144}}) \quad , \qquad (6.1)
\end{gathered}
$$

where $M = 1,\dots,10$ is a vector index of SO(5, 5) and $\alpha, \dot\alpha = 1,\dots,16$ denote the two spinor representations of SO(5, 5). The scalar generators R^{MN} and the 4-form generators

$R^{a_1a_2a_3a_4,MN}$ are antisymmetric in MN and thus belong to the **45** of SO(5,5). Note that the 1-form generators belong to the $\overline{\mathbf{16}}$ of SO(5,5), which is denoted by the $\dot\alpha$ index, as they belong to the representation complex conjugate to the one of the vector fields. The 2-forms belong to the **10**, the 3-forms belong to the **16** and the 5-forms to the $\overline{\mathbf{144}}$.

It is useful to list the conventions for the SO(5,5) Gamma matrices that we are using. In particular, we are using a Weyl basis, so that the Gamma matrices have the form

$$\Gamma_{M,A}{}^{B} = \begin{pmatrix} 0 & \Gamma_{M,\alpha}{}^{\dot\beta} \\ \Gamma_{M,\dot\alpha}{}^{\beta} & 0 \end{pmatrix} \quad , \tag{6.2}$$

where $A = 1, \ldots, 32$. They satisfy the Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} \tag{6.3}$$

where η_{MN} is the Minkowski metric. The charge conjugation matrix is

$$C^{AB} = \begin{pmatrix} 0 & C^{\alpha\dot\beta} \\ C^{\dot\alpha\beta} & 0 \end{pmatrix} \quad , \tag{6.4}$$

which is antisymmetric and unitary, that is

$$C^{\alpha\dot\beta} = -C^{\dot\beta\alpha} \tag{6.5}$$

and

$$C^{\dagger}_{\alpha\dot\beta}C^{\dot\beta\gamma} = \delta^{\gamma}_{\alpha} \qquad C^{\dagger}_{\dot\alpha\beta}C^{\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha} \quad , \tag{6.6}$$

and satisfies the property

$$C\Gamma_M C^{\dagger} = -\Gamma_M^T \quad . \tag{6.7}$$

In this section we will make use of various Fierz identities, the most relevant being

$$(C\Gamma_M)^{(\alpha\beta}\Gamma^{M}{}_{\dot\alpha}{}^{\gamma)} = 0 \qquad (C\Gamma_M)^{(\dot\alpha\dot\beta}\Gamma^{M}{}_{\alpha}{}^{\dot\gamma)} = 0 \quad , \tag{6.8}$$

which is the well-known identity of Gamma matrices in ten dimensions. The 5-form generators satisfy the constraint

$$R^{a_1\ldots a_5,M\alpha}\Gamma_{M\alpha}{}^{\dot\alpha} = 0 \quad , \tag{6.9}$$

which indeed restricts them in the $\overline{\mathbf{144}}$ of SO(5,5).

We now analyse the commutation relations. The SO(5,5) algebra is

$$[R^{MN}, R^{PQ}] = \eta^{MP}R^{NQ} - \eta^{NP}R^{MQ} + \eta^{NQ}R^{MP} - \eta^{MQ}R^{NP} \tag{6.10}$$

while the commutators of the SO(5,5) generators is

$$\begin{aligned} [R^{MN}, R^{a,\dot\alpha}] &= -\frac{1}{2}\Gamma^{MN}{}_{\dot\beta}{}^{\dot\alpha}R^{a,\dot\beta} \\ [R^{MN}, R^{ab,P}] &= \eta^{MP}R^{ab,N} - \eta^{NP}R^{ab,M} \\ [R^{MN}, R^{abc,\alpha}] &= -\frac{1}{2}\Gamma^{MN}{}_{\beta}{}^{\alpha}R^{abc,\beta} \end{aligned} \tag{6.11}$$

and similarly for the higher rank generators. The commutators of the positive level generators of eq. (6.1) are

$$\begin{aligned}
[R^{a_1,\dot\alpha}, R^{a_2,\dot\beta}] &= (C\Gamma_M)^{\dot\alpha\dot\beta} R^{a_1a_2,M} \\
[R^{a_1,\dot\alpha}, R^{a_2a_3,M}] &= \Gamma^M{}_\alpha{}^{\dot\alpha} R^{a_1a_2a_3,\alpha} \\
[R^{a_1a_2,M}, R^{a_3a_4,N}] &= R^{a_1\dots a_4,MN} \\
[R^{a_1,\dot\alpha}, R^{a_2a_3a_4,\alpha}] &= \frac{1}{4}(C\Gamma_{MN})^{\dot\alpha\alpha} R^{a_1\dots a_4,MN} \\
[R^{a_1,\dot\alpha}, R^{a_2\dots a_5,MN}] &= \Gamma^{[M}{}_\alpha{}^{\dot\alpha} R^{a_1\dots a_5,N]\alpha} \\
[R^{a_1a_2,M}, R^{a_3a_4a_5,\alpha}] &= -\frac{1}{2} R^{a_1\dots a_5,M\alpha} \quad .
\end{aligned} \tag{6.12}$$

One can show that all the Jacobi identities are satisfied. From this algebra, one can write down the group element

$$\begin{aligned}
g = e^{x\cdot P} e^{A_{a_1\dots a_5,M\alpha}R^{a_1\dots a_5,M\alpha}} e^{A_{a_1\dots a_4,MN}R^{a_1\dots a_4,MN}} e^{A_{a_1a_2a_3,\alpha}R^{a_1a_2a_3,\alpha}} \\
e^{A_{a_1a_2,M}R^{a_1a_2,M}} e^{A_{a,\dot\alpha}R^{a,\dot\alpha}} e^{\phi_{MN}R^{MN}}
\end{aligned} \tag{6.13}$$

and compute the Maurer-Cartan form. The field strengths of the massless theory are then obtained antisymmetrising the spacetime indices of the various terms in the Maurer-Cartan form, and the field equations of the supergravity theory arise as duality relations. In particular, the field-strength of the vector is dual to the field-strength of the 3-form, while the field strength of the 4-form is dual to the derivative of the scalars. The 2-forms satisfy self-duality conditions, while the field strength of the 5-form vanishes in the massless theory. In deriving the field strengths of the massless theory one takes the positive level E_{11} generators to commute with momentum. In the following we will consider the deformation of the E_{11} algebra which results from modifying the commutation relations of the E_{11} generators with momentum compatibly with the Jacobi identities. As already shown in other cases these deformations exactly coincide with all the possible massive deformations of the corresponding supergravity theory.

We thus consider all the consistent deformations of the massless algebra. Restricting the general analysis of section 2 to the particular case of six dimensions, we write the most general commutators of the first three positive level E_{11} generators with momentum as

$$\begin{aligned}
[R^{a,\dot\alpha}, P_b] &= -g\Theta^{\dot\alpha,MN} R_{MN} \\
[R^{a_1a_2,M}, P_b] &= -gW^M_{(2)\dot\alpha}\delta_b^{[a_1} R^{a_2],\dot\alpha} \\
[R^{a_1a_2a_3,\alpha}, P_b] &= -gW^\alpha_{(3)M}\delta_b^{[a_1} R^{a_2a_3],M} \quad ,
\end{aligned} \tag{6.14}$$

where Θ is antisymmetric in MN. It turns out that the Jacobi identities involving these operators are enough to restrict the representation of Θ completely and to determine $W_{(2)}$ and $W_{(3)}$ uniquely in terms of Θ. This is what we are showing now. As done already in other sections for different dimensions, we can assume that the upstairs spacetime indices are all antisymmetrised when we compute the Jacobi identities. Indeed, the terms which are not completely antisymmetric are cancelled by deforming the E_{11} commutation relations in terms of Og 1 operators. The details of this mechanism were shown in [17] for various examples. In this paper we are only interested in the part of the algebra which is relevant for the determination of the field strengths, and thus we do not need to determine the part of the deformation which involves the Og generators.

The Jacobi identity between two 1-forms and momentum gives the relation

$$(C\Gamma)^{\dot\alpha\dot\beta} W^M_{(2)\dot\gamma} = -\frac{1}{2}\Gamma_{MN,\dot\gamma}{}^{\dot\alpha}\Theta^{\dot\beta,MN} - \frac{1}{2}\Gamma_{MN,\dot\gamma}{}^{\dot\beta}\Theta^{\dot\alpha,MN} \quad , \tag{6.15}$$

while the Jacobi identity between the 1-form, the 2-form and momentum gives

$$\Gamma^{M\dot\alpha}_{\alpha} W^{\alpha,N}_{(3)} = 2\Theta^{\dot\alpha,MN} - W^M_{(2)\dot\beta}(C\Gamma^N)^{\dot\alpha\dot\beta} \tag{6.16}$$

The antisymmetry of Θ in MN in the last equation implies

$$W^{\alpha,M}_{(3)} = C^{\alpha\dot\alpha} W^M_{(2)\dot\alpha} \quad , \tag{6.17}$$

as can be shown taking the part of eq. (6.16) which is symmetric in MN, and therefore does not contain Θ, and suitably contracting with Gamma matrices. Eq. (6.16) thus becomes

$$\Theta^{\dot\alpha,MN} = -(C\Gamma^{[M})^{\dot\alpha\dot\beta} W^{N]}_{(2)\dot\beta} \quad , \tag{6.18}$$

and substituting this back in eq. (6.15) gives

$$[(C\Gamma_M)^{\dot\alpha\dot\beta}\delta^{\dot\delta}_{\dot\gamma} + \frac{1}{2}(C\Gamma^N)^{\dot\beta\dot\delta}\Gamma_{MN,\dot\gamma}{}^{\dot\alpha} + \frac{1}{2}(C\Gamma^N)^{\dot\alpha\dot\delta}\Gamma_{MN,\dot\gamma}{}^{\dot\beta}] W^M_{(2)\dot\delta} = 0 \quad . \tag{6.19}$$

Using the Fierz identity of eq. (6.8) one can show that this equation implies

$$\Gamma_{M,\alpha}{}^{\dot\alpha} W^M_{(2)\dot\alpha} = 0 \quad . \tag{6.20}$$

This analysis shows that the most general deformation of the algebra is encoded in the embedding tensor

$$\Theta^M_{\dot\alpha} = -W^M_{(2)\dot\alpha} \tag{6.21}$$

which belongs to the $\overline{\mathbf{144}}$. This is exactly the embedding tensor of the maximal supergravity theory in six dimensions [8], and this analysis shows again, as in any dimension, that the linear (or representation) constraint of the embedding tensor is completely encoded in the (deformed) E_{11} algebra.

One can determine the commutation relation of the 4-form and the 5-form with momentum requiring that all the Jacobi identities close. The final result is

$$\begin{aligned}
[R^{a,\dot\alpha}, P_b] &= -g(C\Gamma^M)^{\dot\alpha\dot\beta}\Theta^N_{\dot\beta}\delta^a_b R_{MN} \\
[R^{a_1a_2,M}, P_b] &= g\Theta^M_{\dot\alpha}\delta^{[a_1}_b R^{a_2],\dot\alpha} \\
[R^{a_1a_2a_3,\alpha}, P_b] &= gC^{\alpha\dot\alpha}\Theta^M_{\dot\alpha}\delta^{[a_1}_b R^{a_2a_3]}{}_M \\
[R^{a_1\dots a_4,MN}, P_b] &= -2g\Gamma^{[M\dot\alpha}_{\alpha}\Theta^{N]}_{\dot\alpha}\delta^{[a_1}_b R^{a_2a_3a_4],\alpha} \\
[R^{a_1\dots a_5,M\alpha}, P_b] &= -2gC^{\alpha\dot\alpha}\Theta_{N\dot\alpha}\delta^{[a_1}_b R^{a_2\dots a_5],MN} - \frac{g}{2}(C\Gamma_{NP})^{\dot\alpha\alpha}\Theta^M_{\dot\alpha}\delta^{[a_1}_b R^{a_2\dots a_5],NP}
\end{aligned} \tag{6.22}$$

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All the quadratic constraints on the embedding tensor result from the Jacobi identities involving a positive level E_{11} generator and two momentum operators, as well as the Jacobi identities involving a positive level E_{11} generator, the momentum operator and the scalar operator $R_{MN}\Theta^M_{\dot\alpha}$.

It is straightforward to compute the field strengths from the algebra above, using the general results of section 2 and appendix B. The field strength of the vectors is

$$F_{a_1a_2,\dot\alpha} = 2\left[\partial_{[a_1}A_{a_2],\dot\alpha} - g\Theta^M_{\dot\alpha}A_{a_1a_2,M} + \frac{g}{4}\Theta^M_{\dot\gamma}(C\Gamma^N)^{\dot\beta\dot\gamma}\Gamma_{MN,\dot\alpha}{}^{\dot\delta}A_{[a_1,\dot\beta}A_{a_2],\dot\delta}\right] \quad , \tag{6.23}$$

the field strength of the 2-form is

$$\begin{aligned} F_{a_1a_2a_3,M} =3\Big[&\partial_{[a_1}A_{a_2a_3,M} + \frac{1}{2}(C\Gamma_M)^{\dot\alpha\dot\beta}\partial_{[a_1}A_{a_2,\dot\alpha}A_{a_3],\dot\beta} - gC^{\alpha\dot\alpha}\Theta_{M\dot\alpha}A_{a_1a_2a_3,\alpha} \\ &-g(C\Gamma_M)^{\dot\alpha\dot\beta}\Theta^N_{\dot\beta}A_{[a_1a_2,N}A_{a_3],\dot\alpha} + \frac{g}{12}(C\Gamma_{MNP})^{\dot\beta\dot\gamma}(C\Gamma^N)^{\dot\alpha\dot\delta}\Theta^P_{\dot\delta}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3],\dot\gamma}\Big], \end{aligned} \tag{6.24}$$

the field strength of the 3-form is

$$\begin{aligned} F_{a_1\ldots a_4,\alpha} = 4\Big[&\partial_{[a_1}A_{a_2\ldots a_4],\alpha} - \Gamma^{M\dot\alpha}_{\alpha}\partial_{[a_1}A_{a_2a_3,M}A_{a_4],\dot\alpha} \\ &-\frac{1}{6}(C\Gamma_M)^{\dot\alpha\dot\beta}\Gamma^{M\dot\gamma}_{\alpha}\partial_{[a_1}A_{a_2,\dot\alpha}A_{a_3,\dot\beta}A_{a_4],\dot\gamma} + 2g\Gamma^{M\dot\alpha}_{\alpha}\Theta^N_{\dot\alpha}A_{a_1\ldots a_4,MN} \\ &+gC^{\beta\dot\beta}\Theta^M_{\dot\beta}\Gamma_{M,\alpha}{}^{\dot\alpha}A_{[a_1a_2a_3,\beta}A_{a_4],\dot\alpha} - \frac{g}{2}\Theta^M_{\dot\alpha}\Gamma^{N\dot\alpha}_{\alpha}A_{[a_1a_2,M}A_{a_3a_4],N} \\ &+\frac{g}{2}\Theta^M_{\dot\gamma}(C\Gamma_N)^{\dot\alpha\dot\gamma}\Gamma^{N\dot\beta}_{\alpha}A_{[a_1a_2,M}A_{a_3,\dot\alpha}A_{a_4],\dot\beta} \\ &-\frac{g}{48}(C\Gamma^M)^{\dot\alpha\dot\epsilon}\Theta^N_{\dot\epsilon}(C\Gamma_{MNP})^{\dot\beta\dot\gamma}\Gamma^{P\dot\delta}_{\alpha}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3,\dot\gamma}A_{a_4],\dot\delta}\Big] \end{aligned} \tag{6.25}$$

and the field strength of the 4-form is

$$\begin{aligned} F_{a_1\ldots a_5,MN} = 5\Big[&\partial_{[a_1}A_{a_2\ldots a_5],MN} + \frac{1}{4}(C\Gamma_{MN})^{\dot\alpha\alpha}A_{[a_1,\dot\alpha}\partial_{a_2}A_{a_3a_4a_5],\alpha} \\ &-\frac{1}{2}A_{[a_1a_2,[M}\partial_{a_3}A_{a_4a_5],N]} + \frac{1}{8}(C\Gamma_{MNP})^{\dot\alpha\dot\beta}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}\partial_{a_3}A^P_{a_4a_5]} \\ &+\frac{1}{4\cdot 4!}(C\Gamma_{MNP})^{\dot\alpha\dot\beta}(C\Gamma^P)^{\dot\gamma\dot\delta}A_{[a_1,\dot\alpha}A_{a_2\dot\beta}A_{a_3,\dot\gamma}\partial_{a_4}A_{a_5],\dot\delta} \\ &+2gA_{a_1\ldots a_5,[M\alpha}C^{\alpha\dot\alpha}\Theta_{N]\dot\alpha} + \frac{g}{2}A_{a_1\ldots a_5,P\alpha}(C\Gamma_{MN})^{\dot\alpha\alpha}\Theta^P_{\dot\alpha} \\ &+\frac{g}{2}(C\Gamma_{MN})^{\dot\alpha\alpha}\Gamma^{P\dot\beta}_{\alpha}\Theta^Q_{\dot\beta}A_{[a_1,\dot\alpha}A_{a_2\ldots a_5],PQ} - gC^{\alpha\dot\alpha}\Theta_{[M\dot\alpha}A_{[a_1a_2,N]}A_{a_3a_4a_5],\alpha} \\ &-\frac{g}{8}(C\Gamma_{MNP})^{\dot\alpha\dot\beta}C^{\alpha\dot\gamma}\Theta^P_{\dot\gamma}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3a_4a_5],\alpha} \\ &-\frac{g}{8}(C\Gamma_{MN})^{\dot\alpha\alpha}\Gamma^{P\dot\beta}_{\alpha}\Theta^Q_{\dot\beta}A_{[a_1,\dot\alpha}A_{a_2a_3,P}A_{a_4a_5],Q} \\ &-\frac{g}{4\cdot 3!}(C\Gamma_{MNQ})^{\dot\alpha\dot\beta}(C\Gamma^Q)^{\dot\gamma\dot\delta}\Theta^P_{\dot\delta}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3,\dot\gamma}A_{a_4a_5],P} \\ &+\frac{g}{8\cdot 5!}(C\Gamma_{MNR})^{\dot\alpha\dot\beta}(C\Gamma^R{}_{PQ})^{\dot\gamma\dot\delta}(C\Gamma^P)^{\dot\epsilon\dot\rho}\Theta^Q_{\dot\rho}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3,\dot\gamma}A_{a_4,\dot\delta}A_{a_5],\dot\epsilon}\Big]. \end{aligned} \tag{6.26}$$

Using the general results of section 2 that are summarised in appendix B we also determine the gauge transformations of the fields under which the field strengths above transform covariantly. The gauge transformation of the 1-form is

$$\delta A_{a,\dot\alpha} = a_{a,\dot\alpha} - \frac{1}{2} a^{MN} \Gamma_{MN,\dot\alpha}{}^{\dot\beta} A_{a,\dot\beta} \quad , \tag{6.27}$$

the gauge transformation of the 2-form is

$$\delta A_{a_1a_2,M} = a_{a_1a_2,M} - \frac{1}{2}(C\Gamma_M)^{\dot\alpha\dot\beta} A_{[a_1,\dot\alpha} a_{a_2],\dot\beta} + 2a_M{}^N A_{a_1a_2,N} \quad , \tag{6.28}$$

the gauge transformation of the 3-form is

$$\begin{aligned}\delta A_{a_1a_2a_3,\alpha} = {} & a_{a_1a_2a_3,\alpha} + \Gamma^{M\dot\alpha}_{\alpha} A_{[a_1a_2,M} a_{a_3],\dot\alpha} - \frac{1}{3!}(C\Gamma_M)^{\dot\beta\dot\gamma}\Gamma^{M\dot\alpha}_{\alpha} A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}a_{a_3],\dot\gamma} \\ & -\frac{1}{2}a^{MN}\Gamma_{MN,\alpha}{}^{\beta} A_{a_1a_2a_3,\beta} \quad ,\end{aligned} \tag{6.29}$$

the gauge transformation of the 4-form is

$$\begin{aligned}\delta A_{a_1\ldots a_4,MN} = {} & a_{a_1\ldots a_4,MN} - \frac{1}{2}A_{[a_1a_2,[M}a_{a_3a_4],N]} - \frac{1}{4}(C\Gamma_{MN})^{\dot\alpha\alpha}A_{[a_1a_2a_3,\alpha}a_{a_4],\dot\alpha} \\ & -\frac{1}{4\cdot 4!}(C\Gamma_{MNP})^{\dot\alpha\dot\beta}(C\Gamma^P)^{\dot\gamma\dot\delta}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3,\dot\gamma}a_{a_4],\dot\delta} \\ & +\frac{1}{4}(C\Gamma_{[M})^{\dot\alpha\dot\beta}A_{[a_1a_2,N]}A_{a_3,\dot\alpha}a_{a_4],\dot\beta} + 4a_{[M}{}^{P}A_{a_1\ldots a_4,|P|N]}\end{aligned} \tag{6.30}$$

and the gauge transformation of the 5-form is

$$\begin{aligned}\delta A_{a_1\ldots a_5,M\alpha} = {} & \partial_{[a_1}\Lambda_{a_2\ldots a_5],M\alpha} - \frac{1}{2}A_{[a_1a_2a_3,\alpha}a_{a_4a_5],M} - \Gamma^{N\dot\alpha}_{\alpha}A_{[a_1\ldots a_4,MN}a_{a_5],\dot\alpha} \\ & -\frac{1}{4}\Gamma^{N\dot\alpha}_{\alpha}A_{[a_1a_2,M}A_{a_3a_4,N}a_{a_5],\dot\alpha} + \frac{1}{2\cdot 3!}(C\Gamma_N)^{\dot\beta\dot\gamma}\Gamma^{N\dot\alpha}_{\alpha}A_{[a_1a_2,M}A_{a_3,\dot\alpha}A_{a_4,\dot\beta}a_{a_5],\dot\gamma} \\ & +\frac{1}{4\cdot 5!}(C\Gamma_{MNP})^{\dot\beta\dot\alpha}\Gamma^{N\dot\alpha}_{\alpha}(C\Gamma^P)^{\dot\delta\dot\epsilon}A_{[a_1,\dot\alpha}A_{a_2,\dot\beta}A_{a_3,\dot\gamma}A_{a_4,\dot\delta}a_{a_5],\dot\epsilon} \\ & -\frac{1}{2}a^{NP}\Gamma_{NP,\alpha}{}^{\beta}A_{a_1\ldots a_5,M\beta} + 2a_M{}^N A_{a_1\ldots a_5,N\alpha} \quad ,\end{aligned} \tag{6.31}$$

where the parameters a are given in terms of the gauge parameters Λ according to eqs. (2.52) and (2.55), which in the six-dimensional case are

$$\begin{aligned}a_{MN} &= -g\Lambda_{\dot\alpha}(C\Gamma_{[M})^{\dot\alpha\dot\beta}\Theta_{N]\dot\beta} \\ a_{a,\dot\alpha} &= \partial_a\Lambda_{\dot\alpha} + g\Theta^M_{\dot\alpha}\Lambda_{a,M} \\ a_{a_1a_2,M} &= \partial_{[a_1}\Lambda_{a_2],M} + gC^{\alpha\dot\alpha}\Theta_{M\dot\alpha}\Lambda_{a_1a_2,\alpha} \\ a_{a_1a_2a_3,\alpha} &= \partial_{[a_1}\Lambda_{a_2a_3],\alpha} - 2g\Gamma^{M\dot\alpha}_{\alpha}\Theta^N_{\dot\alpha}\Lambda_{a_1a_2a_3,MN} \\ a_{a_1\ldots a_4,MN} &= \partial_{[a_1}\Lambda_{a_2a_3a_4],MN} - (2gC^{\alpha\dot\alpha}\delta^P_{[M}\Theta_{N]\dot\alpha} + \frac{g}{2}(C\Gamma_{MN})^{\dot\alpha\alpha}\Theta^P_{\dot\alpha})\Lambda_{a_1\ldots a_4,P\alpha} \,.\end{aligned} \tag{6.32}$$

One can easily determine also the field strength of the 5-form (up to the term containing the 6-form potential) using the formulae in appendix B.

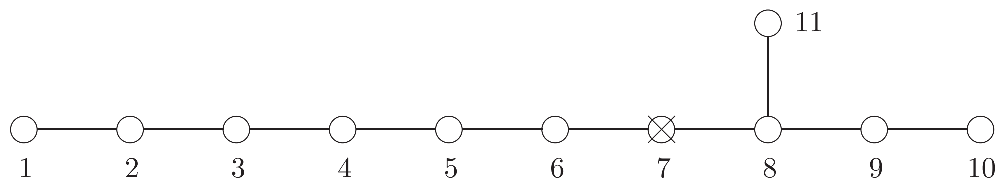

Figure 5. The E_{11} Dynkin diagram corresponding to 7-dimensional supergravity. The internal symmetry group is $\mathrm{SL}(5,\mathbb{R})$.

7 D=7

The multiplet describing massless maximal supergravity theory in 7 dimensions [26] has a bosonic sector containing 14 scalars parametrising $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)$, the metric, a 1-form in the $\overline{\mathbf{10}}$ and a 2-form in the $\mathbf{5}$ of $\mathrm{SL}(5,\mathbb{R})$. This theory results from E_{11} after deletion of node 7, as shown in the Dynkin diagram of figure 5. One can see from the diagram that the 1-forms carry two antisymmetric indices of $\mathrm{SL}(5,\mathbb{R})$.

The positive-level E_{11} generators with completely antisymmetric spacetime indices and up to the 6-form included are

$$
\begin{gathered}
R^M{}_N \ \ (\mathbf{24}) \quad R^{a,MN} \ \ (\mathbf{10}) \qquad R^{a_1a_2}{}_M \ \ (\overline{\mathbf{5}}) \quad R^{a_1a_2a_3,M} \ \ (\mathbf{5}) \qquad R^{a_1a_2a_3a_4}{}_{MN} \ \ (\overline{\mathbf{10}}) \\
R^{a_1\ldots a_5,M}{}_N \ \ (\mathbf{24}) \qquad R^{a_1\ldots a_6}{}_{MN,P} \ \ (\mathbf{40}) \qquad R^{a_1\ldots a_6,MN} \ \ (\mathbf{15}) \quad ,
\end{gathered}
\tag{7.1}
$$

where $M = 1,\ldots,5$. The scalar generators and the 5-forms are in the adjoint of $\mathrm{SL}(5,\mathbb{R})$ and thus satisfy $R^M{}_M = R^{a_1\ldots a_5,M}{}_M = 0$. The 1-form and the 4-form are antisymmetric in MN. The 6-form $R^{a_1\ldots a_6}{}_{MN,P}$ is antisymmetric in MN and satisfies $R^{a_1\ldots a_6}{}_{[MN,P]} = 0$, which corresponds to the $\mathbf{40}$ of $\mathrm{SL}(5,\mathbb{R})$. Finally, the 6-form $R^{a_1\ldots a_6,MN}$ is symmetric in MN, corresponding to the $\mathbf{15}$ of $\mathrm{SL}(5,\mathbb{R})$.

The scalars generate $\mathrm{SL}(5,\mathbb{R})$,

$$
[R^M{}_N, R^P{}_Q] = \delta^P_N R^M{}_Q - \delta^M_Q R^P{}_N \quad , \tag{7.2}
$$

while the commutators of the other generators with the scalars are

$$
\begin{aligned}
[R^M{}_N, R^{a,PQ}] &= \delta^P_N R^{a,MQ} + \delta^Q_N R^{a,PM} - \frac{2}{5}\delta^M_N R^{a,PQ} \\
[R^M{}_N, R^{ab}{}_P] &= -\delta^M_P R^{ab}{}_N + \frac{1}{5}\delta^M_N R^{ab}{}_P
\end{aligned}
\tag{7.3}
$$

and similarly for the higher level generators.

The commutators of the positive level generators in eq. (7.1) are

$$\begin{aligned}
[R^{a_1,MN}, R^{a_2,PQ}] &= \epsilon^{MNPQR} R^{a_1a_2}{}_R \\
[R^{a_1,MN}, R^{a_2a_3}{}_P] &= \delta_P^{[M} R^{a_1a_2a_3,N]} \\
[R^{a_1a_3}{}_M, R^{a_3a_4}{}_N] &= R^{a_1\dots a_4}{}_{MN} \\
[R^{a_1,MN}, R^{a_2a_3a_4,P}] &= \epsilon^{MNPQR} R^{a_1\dots a_4}{}_{QR} \\
[R^{a_1a_2}{}_M, R^{a_3a_4a_5,N}] &= R^{a_1\dots a_5,N}{}_M \\
[R^{a_1,MN}, R^{a_2\dots a_5}{}_{PQ}] &= -2\delta_{[P}^{[M} R^{a_1\dots a_5,N]}{}_{Q]} \\
[R^{a_1a_2a_3,M}, R^{a_4a_5a_6,N}] &= R^{a_1\dots a_6,MN} \\
[R^{a_1a_2}{}_M, R^{a_3\dots a_6}{}_{NP}] &= R^{a_1\dots a_6}{}_{NP,M} \\
[R^{a_1,MN}, R^{a_2\dots a_6,Q}{}_P] &= \epsilon^{MNQRS} R^{a_1\dots a_6}{}_{RS,P} + \delta_P^{[M} R^{a_1\dots a_6,N]Q} \quad .
\end{aligned} \tag{7.4}$$

To prove that all the Jacobi identities close one makes use of the identity

$$\epsilon^{M_1\dots M_5}\epsilon_{N_1\dots N_5} = 5!\delta_{[N_1\dots N_5]}^{[M_1\dots M_5]} \quad . \tag{7.5}$$

If one considers the group element

$$\begin{aligned}
g = e^{x\cdot P} e^{A_{a_1\dots a_6,MN}R^{a_1\dots a_6,MN}} e^{A^{MN,P}_{a_1\dots a_6}R^{a_1\dots a_6}{}_{MN,P}} e^{A_{a_1\dots a_5,M}{}^N R^{a_1\dots a_5,M}{}_N} e^{A^{MN}_{a_1\dots a_4}R^{a_1\dots a_4}{}_{MN}} \\
e^{A_{a_1a_2a_3,M}R^{a_1a_2a_3,M}} e^{A^M_{a_1a_2}R^{a_1a_2}_M} e^{A_{a,MN}R^{a,MN}} e^{\phi_M{}^N R^M{}_N}
\end{aligned} \tag{7.6}$$

and compute the Maurer-Cartan form using the fact that the positive level generators commute with momentum, the field strengths of the massless theory are obtained antisymmetrising the spacetime indices of the various terms in the Maurer-Cartan form. Requiring that the field strengths satisfy duality relations, that is the field-strength of the vector is dual to the field-strength of the 4-form, the field strength of the 2-form is dual to the field strength of the 3-form and the field strength of the 6-form is dual to the derivative of the scalars, one recovers the field equations of the massless supergravity theory. We now consider the deformations of the E_{11} algebra resulting from modifying the commutation relations of the E_{11} generators with momentum compatibly with the Jacobi identities. In this way we will derive all the gauged supergravities in seven dimensions.

Following the results of section 2, the most general deformation that we can write down is

$$\begin{aligned}
[R^{a,MN}, P_b] &= -g\Theta^{MN,P}{}_Q R^Q{}_P \\
[R^{a_1a_2}{}_M, P_b] &= -gW^{(2)}_{M,NP}\delta_b^{[a_1} R^{a_2],NP} \\
[R^{a_1a_2a_3,M}, P_b] &= -gW^{M,N}_{(3)}\delta_b^{[a_1} R^{a_2a_3]}{}_N \quad ,
\end{aligned} \tag{7.7}$$

where $\Theta^{MN,P}{}_Q$ is antisymmetric in MN and $\Theta^{MN,P}{}_P = 0$, and $W^{(2)}_{M,NP}$ is antisymmetric in NP. For simplicity we have only considered the E_{11} generators up to level 4. Indeed it turns out that the constraints that the Jacobi identities impose on Θ, $W_{(2)}$ and $W_{(3)}$ are enough to restrict the representations of Θ uniquely.

The Jacobi identity involving two 1-forms and the momentum operator gives

$$\Theta^{PQ,[M}{}_{[S}\delta^{N]}_{T]} + \Theta^{MN,[P}{}_{[S}\delta^{Q]}_{T]} = \frac{1}{2}\epsilon^{MNPQR}W^{(2)}_{R,ST} \quad , \tag{7.8}$$

which can be solved for $W^{(2)}$ in terms of Θ giving

$$W^{(2)}_{R,ST} = \epsilon_{MTPQR}\Theta^{PQ,M}{}_S - \epsilon_{MSPQR}\Theta^{PQ,M}{}_T \quad . \tag{7.9}$$

The Jacobi identity involving the 1-form, the 2-form and the momentum operator gives

$$W^{(2)}_{P,RS}\epsilon^{MNQRS} + \Theta^{MN,Q}{}_P = -\frac{1}{2}W^{N,Q}_{(3)}\delta^M_P + \frac{1}{2}W^{M,Q}_{(3)}\delta^N_P \quad . \tag{7.10}$$

We can analyse the solutions of eqs. (7.9) and (7.10) for different representations of Θ. If we take $\Theta^{MN,P}{}_Q$ such that $\Theta^{[MN,P]}{}_Q = 0$, then eq. (7.9) implies

$$W^{(2)}_{M,NP} = 0 \quad . \tag{7.11}$$

Substituting this in eq. (7.10) and using $\Theta^{[MN,P]}{}_Q = 0$ and $\Theta^{MN,P}{}_P = 0$ one obtains that $W^{M,N}_{(3)}$ is symmetric in MN. One thus obtains the embedding tensor $\Theta^{MN} = W^{MN}_{(3)}$ in the **15** of $\mathrm{SL}(5,\mathbb{R})$ and we will then show that the inclusion of the forms of rank higher that 3 is also compatible with this deformation. If we instead take $\Theta^{MN,P}{}_Q$ to be completely antisymmetric in MNP, then we can write

$$\Theta^{MN,P}{}_Q = \epsilon^{MNPRS}\Theta_{RS,Q} \quad , \tag{7.12}$$

and the condition $\Theta^{MN,P}{}_P = 0$ implies $\Theta_{[MN,Q]} = 0$. Therefore the embedding tensor $\Theta_{MN,P}$ belongs to the **40** of $\mathrm{SL}(5,\mathbb{R})$. Eq. (7.9) then gives

$$W^{(2)}_{M,NP} = -\Theta_{NP,M} \quad , \tag{7.13}$$

and eq. (7.10) gives

$$W^{M,N}_{(3)} = 0 \quad . \tag{7.14}$$

Also in this case we will show that one can consistently include in the algebra the higher rank form generators. We now proceed with the analysis of the algebra and the derivation of the field strengths and gauge transformations for the two different cases corresponding to an embedding tensor in the **15** and in the **40**. One can show that a linear combination of these two deformations is not allowed because of the quadratic constraints.

7.1 Embedding tensor in the 15 of $\mathrm{SL}(5,\mathbb{R})$

We now determine the commutation relations or all the generators in eq. (7.1) with the momentum operator requiring the closure of the Jacobi identities. In the case of the embedding tensor Θ^{MN} in the **15** we get

$$\begin{aligned}
[R^{a,MN},P_b] &= -g\Theta^{[M|P|}\delta^a_b R^{N]}{}_P \\
[R^{a_1a_2}{}_M,P_b] &= 0 \\
[R^{a_1a_2a_3,M},P_b] &= -g\Theta^{MN}\delta^{[a_1}_b R^{a_2a_3]}{}_N \\
[R^{a_1a_2a_3a_4}{}_{MN},P_b] &= 0 \\
[R^{a_1\dots a_5,M}{}_N,P_b] &= g\Theta^{MP}\delta^{[a_1}_b R^{a_2\dots a_5]}{}_{PN} \\
[R^{a_1\dots a_6}{}_{MN,P},P_b] &= 0 \\
[R^{a_1\dots a_6,MN},P_b] &= -2g\Theta^{P(M}\delta^{[a_1}_b R^{a_2\dots a_6,N)}{}_P \quad .
\end{aligned} \tag{7.15}$$

From the algebra above one computes the field strengths using the general results of section 2 and appendix B. The field strength of the vectors is

$$F_{a_1a_2,MN} = 2[\partial_{[a_1}A_{a_2],MN} + g\Theta^{PQ}A_{[a_1,PM}A_{a_2],QN}] \quad , \tag{7.16}$$

the field strength of the 2-form is

$$\begin{aligned}
F^M_{a_1a_2a_3} = 3\Big[&\partial_{[a_1}A^M_{a_2a_3]} + \frac{1}{2}\epsilon^{MNPQR}\partial_{[a_1}A_{a_2,NP}A_{a_3],QR} + g\Theta^{MN}A_{a_1a_2a_3,N} \\
&+\frac{g}{6}\Theta^{NQ}\epsilon^{MPRST}A_{[a_1,NP}A_{a_2,QR}A_{a_3],ST}\Big] \quad ,
\end{aligned} \tag{7.17}$$

the field strength of the 3-form is

$$\begin{aligned}
F_{a_1\dots a_4,M} = 4\Big[&\partial_{[a_1}A_{a_2\dots a_4],M} - \partial_{[a_1}A^N_{a_2a_3}A_{a_4],NM} - \frac{1}{6}\partial_{[a_1}A_{a_2,NP}A_{a_3,QR}A_{a_4],SM}\epsilon^{SNPQR} \\
&-g\Theta^{NP}A_{[a_1a_2a_3,N}A_{a_4],PM} - \frac{g}{12}\Theta^{NQ}\epsilon^{PRSTU}A_{[a_1,NP}A_{a_2,QR}A_{a_3,ST}A_{a_4],UM}\Big]
\end{aligned} \tag{7.18}$$

and the field strength of the 4-form is

$$\begin{aligned}
F^{MN}_{a_1\dots a_5} = 5\Big[&\partial_{[a_1}A^{MN}_{a_2\dots a_5]} + \epsilon^{PQRMN}A_{[a_1,PQ}\partial_{a_2}A_{a_3\dots a_5],R} - \frac{1}{2}A^{[M}_{[a_1a_2}\partial_{a_3}A^{N]}_{a_4a_5]} \\
&+\frac{1}{2}\epsilon^{PQSMN}A_{[a_1,PQ}A_{a_2,RS}\partial_{a_3}A^R_{a_4a_5]} \\
&+\frac{1}{4!}\epsilon^{TUVWR}\epsilon^{PQSMN}A_{[a_1,PQ}A_{a_2,RS}A_{a_3,TU}\partial_{a_4}A_{a_5],VW} \\
&-g\Theta^{P[M}A^{N]}_{a_1\dots a_5} - g\Theta^{P[M}A^{N]}_{[a_1a_2}A_{a_3a_4a_5],P} \\
&+\frac{g}{2}\epsilon^{PQSMN}\Theta^{RT}A_{[a_1,PQ}A_{a_2,RS}A_{a_3a_4a_5],T} \\
&-\frac{g}{2\cdot 5!}\epsilon^{PQSMN}\epsilon^{TUYWR}\Theta^{VX}A_{[a_1,PQ}A_{a_2,RS}A_{a_3,TU}A_{a_4,VW}A_{a_5],XY}\Big].
\end{aligned} \tag{7.19}$$

These field strengths transform covariantly under the gauge transformations

$$
\begin{aligned}
\delta A_{a,MN} &= a_{a,MN} + 2a_{[M}{}^{P} A_{a,|P|N]} \\
\delta A^{M}_{a_1a_2} &= a^{M}_{a_1a_2} - \frac{1}{2}\epsilon^{MNPQR} A_{[a_1,NP} a_{a_2],QR} - a_N{}^{M} A^{N}_{a_1a_2} \\
\delta A_{a_1a_2a_3,M} &= a_{a_1a_2a_3,M} + A^{N}_{[a_1a_2} a_{a_3],NM} + \frac{1}{3!}\epsilon^{NQRST} A_{[a_1,MN} A_{a_2,QR} a_{a_3],ST} \\
&\quad + a_M{}^{N} A_{a_1a_2a_3,N} \\
\delta A^{MN}_{a_1\ldots a_4} &= a^{MN}_{a_1\ldots a_4} - \frac{1}{2} A^{[M}_{[a_1a_2} a^{N]}_{a_3a_4]} - \epsilon^{MNPQR} A_{[a_1a_2a_3,P} a_{a_4],QR} \\
&\quad - \frac{1}{4!}\epsilon^{TUVWR}\epsilon^{PQSMN} A_{[a_1,PQ} A_{a_2,RS} A_{a_3,TU} a_{a_4],VW} \\
&\quad + \frac{1}{4}\epsilon^{QRST[M} A^{N]}_{[a_1a_2} A_{a_3,QR} a_{a_4],ST} \\
\delta A_{a_1\ldots a_5,M}{}^{N} &= \partial_{[a_1}\Lambda_{a_2\ldots a_5],M}{}^{N} - A_{[a_1a_2a_3,M} a^{N}_{a_4a_5]} - 2A^{PN}_{[a_1\ldots a_4} a_{a_5],PM} \\
&\quad + \frac{1}{2} A^{N}_{[a_1a_2} A^{Q}_{a_3a_4} a_{a_5],QM} \\
&\quad + \frac{2}{5!}\epsilon^{VWXYT}\epsilon^{RSUPN} A_{[a_1,PM} A_{a_2,RS} A_{a_3,TU} A_{a_4,VW} a_{a_5],XY} \\
&\quad - \frac{1}{3!}\epsilon^{STUVQ} A^{N}_{[a_1a_2} A_{a_3,QM} A_{a_4,ST} a_{a_5],UV} - 2a_P{}^{[M} A^{|P|N]}_{a_1\ldots a_5} \quad ,
\end{aligned}
\tag{7.20}
$$

where the parameters a are given in terms of the gauge parameters as

$$
\begin{aligned}
a_M{}^{N} &= g\Lambda_{MP}\Theta^{PN} \\
a_{a,MN} &= \partial_a \Lambda_{MN} \\
a^{M}_{a_1a_2} &= \partial_{[a_1}\Lambda^{M}_{a_2]} - g\Theta^{MN}\Lambda_{a_1a_2,N} \\
a_{a_1a_2a_3,M} &= \partial_{[a_1}\Lambda_{a_2a_3],M} \\
a^{MN}_{a_1\ldots a_4} &= \partial_{[a_1}\Lambda^{MN}_{a_2\ldots a_4]} + g\Theta^{P[M}\Lambda_{a_1\ldots a_4.P}{}^{N]} \quad .
\end{aligned}
\tag{7.21}
$$

One can easily determine the field strengths and the gauge transformations of the fields of higher rank using the general results on section 2 which are explicitly expanded in appendix B.

7.2 Embedding tensor in the 40 of $\mathrm{SL}(5,\mathbb{R})$

In the case of the embedding tensor $\Theta_{MN,P}$, which belongs to the **40**, one gets

$$
\begin{aligned}
[R^{a,MN}, P_b] &= -g\epsilon^{MNPQR}\Theta_{PQ,S}\delta^{a}_{b} R^{S}{}_{R} \\
[R^{a_1a_2}{}_{M}, P_b] &= g\Theta_{NP,M}\delta^{[a_1}_{b} R^{a_2],NP} \\
[R^{a_1a_2a_3,M}, P_b] &= 0 \\
[R^{a_1a_2a_3a_4}{}_{MN}, P_b] &= -g\Theta_{MN,P}\delta^{[a_1}_{b} R^{a_2a_3a_4],P} \\
[R^{a_1\ldots a_5,M}{}_{N}, P_b] &= g\epsilon^{MPQRS}\Theta_{PQ,N}\delta^{[a_1}_{b} R^{a_2\ldots a_5]}{}_{RS} \\
[R^{a_1\ldots a_6}{}_{MN,P}, P_b] &= -g\Theta_{NP,Q}\delta^{[a_1}_{b} R^{a_2\ldots a_6],Q}{}_{M} - g\Theta_{NQ,M}\delta^{[a_1}_{b} R^{a_2\ldots a_6],Q}{}_{P} \\
&\quad + g\Theta_{PQ,M}\delta^{[a_1}_{b} R^{a_2\ldots a_6],Q}{}_{N} \\
[R^{a_1\ldots a_6,MN}, P_b] &= 0 \quad .
\end{aligned}
\tag{7.22}
$$

For this deformation the field strength of the vector is

$$F_{a_1a_2,MN} = 2[\partial_{[a_1}A_{a_2],MN} - g\Theta_{MN,P}A^P_{a_1a_2} + g\epsilon^{PQRTU}\Theta_{TU,[M}A_{[a_1,|PQ}A_{a_2],R|N]}] \quad , \tag{7.23}$$

the field strength of the 2-form is

$$\begin{aligned} F^M_{a_1a_2a_3} = 3\Big[& \partial_{[a_1}A^M_{a_2a_3]} + \frac{1}{2}\epsilon^{MNPQR}\partial_{[a_1}A_{a_2,NP}A_{a_3],QR} - g\Theta_{ST,P}\epsilon^{MSTQR}A^P_{[a_1a_2}A_{a_3],QR} \\ & + \frac{g}{3}\epsilon^{PQVWR}\Theta_{VW,N}\epsilon^{MNSTU}A_{[a_1,PQ}A_{a_2,RS}A_{a_3],TU}\Big] \quad , \end{aligned} \tag{7.24}$$

the field strength of the 3-form is

$$\begin{aligned} F_{a_1\ldots a_4,M} = 4\Big[& \partial_{[a_1}A_{a_2\ldots a_4],M} - \partial_{[a_1}A^N_{a_2a_3}A_{a_4],NM} - \frac{1}{6}\partial_{[a_1}A_{a_2,NP}A_{a_3,QR}A_{a_4],SM}\epsilon^{SNPQR} \\ & + g\Theta_{NP,M}A^{NP}_{a_1\ldots a_4} - \frac{g}{2}\Theta_{PM,N}A^N_{[a_1a_2}A^P_{a_3a_4]} \\ & + \frac{g}{2}\Theta_{TU,N}\epsilon^{TUPQR}A^N_{[a_1a_2}A_{a_3,PQ}A_{a_4],RM} \\ & - \frac{g}{12}\epsilon^{NPWXQ}\Theta_{WX,Z}\epsilon^{UZRST}A_{[a_1,NP}A_{a_2,QR}A_{a_3,ST}A_{a_4],UM}\Big] \end{aligned} \tag{7.25}$$

and the field strength of the 4-form is

$$\begin{aligned} F^{MN}_{a_1\ldots a_5} = 5\Big[& \partial_{[a_1}A^{MN}_{a_2\ldots a_5]} + \epsilon^{PQRMN}A_{[a_1,PQ}\partial_{a_2}A_{a_3\ldots a_5],R} - \frac{1}{2}A^{[M}_{[a_1a_2}\partial_{a_3}A^{N]}_{a_4a_5]} \\ & + \frac{1}{2}\epsilon^{PQSMN}A_{[a_1,PQ}A_{a_2,RS}\partial_{a_3}A^R_{a_4a_5]} \\ & + \frac{1}{4!}\epsilon^{TUVWR}\epsilon^{PQSMN}A_{[a_1,PQ}A_{a_2,RS}A_{a_3,TU}\partial_{a_4}A_{a_5],VW} \\ & - g\epsilon^{MNPQS}\Theta_{PQ,T}A_{a_1\ldots a_5,S}{}^T + g\epsilon^{MNPQT}\Theta_{RS,T}A_{[a_1,PQ}A^{RS}_{a_2\ldots a_5]} \\ & - \frac{g}{2}\epsilon^{MNPQT}\Theta_{RT,S}A_{[a_1,PQ}A^R_{a_2a_3}A^S_{a_4a_5]} \\ & - \frac{g}{3!}\epsilon^{MNPQS}\epsilon^{TUWXR}\Theta_{WX,V}A_{[a_1,PQ}A_{a_2,RS}A_{a_3,TU}A^V_{a_4a_5]} \\ & - \frac{2g}{5!}\epsilon^{MNPQS}\epsilon^{TUDWR}\epsilon^{XYABV}\Theta_{AB,D}A_{[a_1,PQ}A_{a_2,RS}A_{a_3,TU}A_{a_4,VW}A_{a_5],XY}\Big] \quad . \end{aligned} \tag{7.26}$$

The field strengths transform covariantly under the gauge transformations of eq. (7.20) where the parameters a are given as

$$\begin{aligned} a_M{}^N &= -g\Lambda_{PQ}\epsilon^{PQRSN}\Theta_{RS,M} \\ a_{a,MN} &= \partial_a\Lambda_{MN} + g\Theta_{MN,P}\Lambda^P_a \\ a^M_{a_1a_2} &= \partial_{[a_1}\Lambda^M_{a_2]} \\ a_{a_1a_2a_3,M} &= \partial_{[a_1}\Lambda_{a_2a_3],M} - g\Theta_{NP,M}\Lambda^{NP}_{a_1a_2a_3} \\ a^{MN}_{a_1\ldots a_4} &= \partial_{[a_1}\Lambda^{MN}_{a_2a_3a_4]} + g\epsilon^{PQRMN}\Theta_{QR,S}\Lambda_{a_1\ldots a_4,P}{}^S \quad . \end{aligned} \tag{7.27}$$

The field strengths and the gauge transformations of the higher rank fields can also easily been determined from the above algebra.

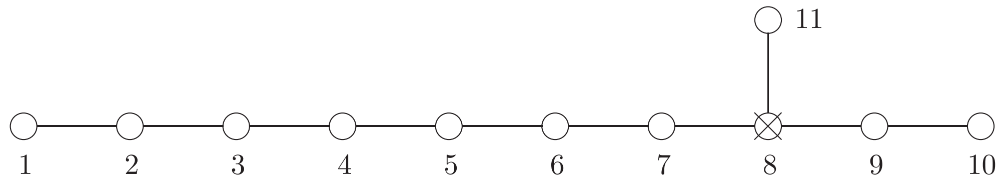

Figure 6. The E_{11} Dynkin diagram corresponding to 8-dimensional supergravity. The internal symmetry group is $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

8 D=8

The bosonic sector of maximal massless eight-dimensional supergravity [27] contains seven scalars parametrising the manifold $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, the metric, a vector in the $(\overline{\mathbf{3}},\mathbf{2})$ of the internal symmetry group $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, a 2-form in $(\mathbf{3},\mathbf{1})$ and an $\mathrm{SL}(2,\mathbb{R})$ doublet of 3-forms which satisfy self-duality conditions. The E_{11} Dynkin diagram corresponding to this theory is shown in figure 6.

The positive-level E_{11} generators with completely antisymmetric spacetime indices and up to the 6-form included, together with their $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ representations, are

$$\begin{gathered} R^i \quad (\mathbf{1},\mathbf{3}) \quad R^M{}_N \quad (\mathbf{8},\mathbf{1}) \quad R^{a,M\alpha} \quad (\mathbf{3},\mathbf{2}) \quad R^{a_1a_2}{}_M \quad (\overline{\mathbf{3}},\mathbf{1}) \quad R^{a_1a_2a_3,\alpha} \quad (\mathbf{1},\mathbf{2}) \\ R^{a_1a_2a_3a_4,M} \quad (\mathbf{3},\mathbf{1}) \quad R^{a_1\dots a_5,\alpha}{}_M \quad (\overline{\mathbf{3}},\mathbf{2}) \quad R^{a_1\dots a_6,i} \quad (\mathbf{1},\mathbf{3}) \quad R^{a_1\dots a_6,M}{}_N \quad (\mathbf{8},\mathbf{1}) \\ R^{a_1\dots a_7,\alpha}{}_{MN} \quad (\overline{\mathbf{6}},\mathbf{2}) \quad R^{a_1\dots a_7,M\alpha} \quad (\mathbf{3},\mathbf{2}) \quad . \end{gathered} \tag{8.1}$$

Here the index $i = 1,2,3$ and $\alpha = 1,2$ denote the adjoint and the fundamental of $\mathrm{SL}(2,\mathbb{R})$ respectively, while the upstairs index $M = 1,2,3$ denotes the fundamental of $\mathrm{SL}(3,\mathbb{R})$. The scalars R^i, $i = 1,2,3$, are the $\mathrm{SL}(2,\mathbb{R})$ generators, while the scalars $R^M{}_N$ are the $\mathrm{SL}(3,\mathbb{R})$ generators and thus satisfy the constraint $R^M{}_M = 0$. The 6-form $R^{a_1\dots a_6,M}{}_N$ also satisfies the constraint $R^{a_1\dots a_6,M}{}_M = 0$, while the 7-form $R^{a_1\dots a_7,\alpha}{}_{MN}$ is symmetric in MN.

The algebra of the scalars is

$$\begin{aligned} [R^i, R^j] &= f^{ij}{}_k R^k \\ [R^M{}_N, R^P{}_Q] &= \delta^P_N R^M{}_Q - \delta^M_Q R^P{}_N \quad , \end{aligned} \tag{8.2}$$

while the other commutation relations with the scalar generators are

$$\begin{aligned} [R^i, R^{a,M\alpha}] &= D^{i}_{\beta}{}^{\alpha} R^{a,M\beta} \\ [R^M{}_N, R^{a,P\alpha}] &= \delta^P_N R^{a,M\alpha} - \frac{1}{3}\delta^M_N R^{a,P\alpha} \end{aligned} \tag{8.3}$$

and similarly for the higher rank forms. Here $D^{i}_{\beta}{}^{\alpha}$ are the generators of $\mathrm{SL}(2,\mathbb{R})$ satisfying

$$[D^i, D^j]_\beta{}^\alpha = f^{ij}{}_k D^{k\alpha}_\beta \tag{8.4}$$

and $f^{ij}{}_k$ are the structure constants of $\mathrm{SL}(2,\mathbb{R})$. In terms of Pauli matrices, a choice of $D^{i}_{\beta}{}^{\alpha}$ is

$$D_1 = \frac{\sigma_1}{2} \qquad D_2 = \frac{i\sigma_2}{2} \qquad D_3 = \frac{\sigma_3}{2} \quad . \tag{8.5}$$

We raise and lower SL(2, $\mathbb{R}$) indices using the antisymmetric metric $\epsilon^{\alpha\beta}$, that is, for a generic doublet V^α,

$$V^\alpha = \epsilon^{\alpha\beta} V_\beta \qquad V_\alpha = V^\beta \epsilon_{\beta\alpha} \quad , \tag{8.6}$$

which implies

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma \quad . \tag{8.7}$$

The generators

$$D^{i,\alpha\beta} = \epsilon^{\alpha\gamma} D^{i}_{\gamma}{}^{\beta} \tag{8.8}$$

are symmetric in $\alpha\beta$. Useful identities relating the SL(2, $\mathbb{R}$) generators are

$$D_i^{\alpha\beta} D^{i,\gamma\delta} = -\frac{1}{4}[\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}] \tag{8.9}$$

and

$$D^{i}_{\beta}{}^{\gamma} D^{j}_{\gamma}{}^{\alpha} + D^{j}_{\beta}{}^{\gamma} D^{i}_{\gamma}{}^{\alpha} = \frac{1}{2} g^{ij}\delta^\alpha_\beta \quad , \tag{8.10}$$

where g^{ij} is the SL(2, $\mathbb{R}$) Killing metric.

The non-vanishing commutation relations involving all the non-scalar generators of eq. (8.1) are

$$\begin{aligned}
[R^{a_1,M\alpha}, R^{a_2,N\beta}] &= \epsilon^{\alpha\beta}\epsilon^{MNP} R^{a_1a_2}{}_P \\
[R^{a_1,M\alpha}, R^{a_2a_3}{}_N] &= \delta^M_N R^{a_1a_2a_3,\alpha} \\
[R^{a_1a_2}{}_M, R^{a_3a_4}{}_N] &= \epsilon_{MNP} R^{a_1a_2a_3a_4,P} \\
[R^{a_1,M\alpha}, R^{a_2a_3a_4,\beta}] &= -\epsilon^{\alpha\beta} R^{a_1a_2a_3a_4,M} \\
[R^{a_1,M\alpha}, R^{a_2\dots a_5,N}] &= \epsilon^{MNP} R^{a_1\dots a_5,\alpha}{}_P \\
[R^{a_1a_2}{}_M, R^{a_3a_4a_5,\alpha}] &= R^{a_1\dots a_5,\alpha}{}_M \\
[R^{a_1,M\alpha}, R^{a_2\dots a_6,\beta}{}_N] &= \epsilon^{\alpha\beta} R^{a_1\dots a_6,M}{}_N + D_i^{\alpha\beta}\delta^M_N R^{a_1\dots a_6,i} \\
[R^{a_1a_2}{}_M, R^{a_3\dots a_6,N}] &= -R^{a_1\dots a_6,N}{}_M \\
[R^{a_1a_2a_3,\alpha}, R^{a_4a_5a_6,\beta}] &= D_i^{\alpha\beta} R^{a_1\dots a_6,i} \\
[R^{a_1,M\alpha}, R^{a_2\dots a_7,i}] &= D^{i}_{\beta}{}^{\alpha} R^{a_1\dots a_7,M\beta} \\
[R^{a_1,M\alpha}, R^{a_2\dots a_7,N}{}_P] &= \frac{3}{8}\delta^M_P R^{a_1\dots a_7,N\alpha} - \frac{1}{8}\delta^N_P R^{a_1\dots a_7,M\alpha} + \epsilon^{MNQ} R^{a_1\dots a_7,\alpha}{}_{PQ} \\
[R^{a_1a_2}{}_M, R^{a_3\dots a_7,\alpha}{}_N] &= \frac{1}{8}\epsilon_{MNP} R^{a_1\dots a_7,P\alpha} - R^{a_1\dots a_7,\alpha}{}_{MN} \\
[R^{a_1a_2a_3,\alpha}, R^{a_4a_5a_6a_7,M}] &= -\frac{1}{4} R^{a_1\dots a_7,M\alpha} \quad .
\end{aligned} \tag{8.11}$$

One can show that all Jacobi identities are satisfied. This requires the use of the identities of eqs. (8.6)–(8.10), as well as the identities

$$\epsilon^{M_1M_2M_3}\epsilon_{N_1N_2N_3} = 6\delta^{[M_1M_2M_3]}_{[N_1N_2N_3]} \tag{8.12}$$

and

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} V^\gamma W^\delta = V^\alpha W^\beta - V^\beta W^\alpha \quad , \tag{8.13}$$

where in the last equation V and W are two generic $\mathrm{SL}(2,\mathbb{R})$ doublets.

The derivation of the field strengths of all the fields and dual fields of massless maximal supergravity follows exactly the same steps as in the other cases. One considers the group element

$$g = e^{x\cdot P} e^{A_{a_1\ldots a_7,M\alpha}R^{a_1\ldots a_7,M\alpha}} \ldots e^{A^M_{a_1a_2}R^{a_1a_2}_M} e^{A_{a,M\alpha}R^{a,M\alpha}} e^{\phi_{MN}R^M{}_N} e^{\phi_i R^i} \quad , \tag{8.14}$$

and computes the Maurer-Cartan form using the fact that in the massless case the positive level generators commute with momentum. In this way one derives the field strengths of the massless theory antisymmetrising the spacetime indices of the various terms in the Maurer-Cartan form. The field equations are then obtained imposing duality conditions for the various field strengths. We now consider all the consistent deformations of the E_{11} algebra resulting from modifying the commutation relations of the E_{11} generators with momentum compatibly with the Jacobi identities. In this way we will derive all the gauged supergravities in eight dimensions.

The representation of the embedding tensor is contained in the tensor product of the representation of the 1-form generator and of the scalar generators. In eight dimensions this leads to

$$(\mathbf{3},\mathbf{2}) \otimes [(\mathbf{1},\mathbf{3}) \oplus (\mathbf{8},\mathbf{1})] = (\mathbf{3},\mathbf{2}) \oplus (\mathbf{3},\mathbf{2}) \oplus (\mathbf{3},\mathbf{4}) \oplus (\overline{\mathbf{6}},\mathbf{2}) \oplus (\mathbf{15},\mathbf{2}) \quad . \tag{8.15}$$

We now show that only including an embedding tensor in the $(\overline{\mathbf{6}},\mathbf{2})$ or in one of the two $(\mathbf{3},\mathbf{2})$ representations leads to a consistent deformation of the algebra.

We first show that the deformations in the $(\mathbf{15},\mathbf{2})$ and in the $(\mathbf{3},\mathbf{4})$ are ruled out. The first case corresponds to the embedding tensor $\Theta_P^{MN,\alpha}$, symmetric in MN and satisfying the traceless condition $\Theta_N^{MN,\alpha} = 0$. We want to write down the commutator

$$[R^{a,M\alpha}, P_b] = -g\Theta_P^{MN,\alpha}\delta_b^a R^P{}_N \quad , \tag{8.16}$$

but the Jacobi identity between $R^{a,M\alpha}$, $R^{b,N\beta}$ and P_c shows that this is ruled out because of symmetry arguments. Analogously, in the $(\mathbf{3},\mathbf{4})$ case we would write

$$[R^{a,M\alpha}, P_b] = -g\Theta^{M,\alpha\beta\gamma}D_{i,\beta\gamma}\delta_b^a R^i \quad , \tag{8.17}$$

where $\Theta^{M,\alpha\beta\gamma}$ is completely symmetric in $\alpha\beta\gamma$, but again the Jacobi identity between $R^{a,M\alpha}$, $R^{b,N\beta}$ and P_c rules this out.

We now consider the two $(\mathbf{3},\mathbf{2})$ deformations. These lead to

$$[R^{a,M\alpha}, P_b] = -g\delta_b^a[a\Theta^{N\alpha}R^M{}_N + b\Theta^{M\beta}D_{i,\beta}{}^\alpha R^i] \tag{8.18}$$

where the parameters a and b are in principle arbitrary, and we now determine the constraints on these parameters that come from the Jacobi identities. The Jacobi identity between $R^{a,M\alpha}$, $R^{b,N\beta}$ and P_c gives

$$b = -\frac{8}{3}a \quad , \tag{8.19}$$

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and we can fix the parameter a to 1. Therefore, only one of the two $(\mathbf{3},\mathbf{2})$ deformations can lead to a consistent algebra. In the remaining of this section we show that this deformation is indeed consistent, and we also show that the embedding tensor in the $(\overline{\mathbf{6}},\mathbf{2})$ leads to a consistent deformation. We do this determining the commutation relations of all the generators in eq. (8.1) with momentum consistently with the Jacobi identities.

8.1 Embedding tensor in the $(\mathbf{3},\mathbf{2})$ of $\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$

We first consider the deformation in the $(\mathbf{3},\mathbf{2})$, that is the embedding tensor $\Theta^{M\alpha}$. The result is

$$\begin{aligned}
[R^{a,M\alpha},P_b] &= -g\delta^a_b[\Theta^{N\alpha}R^M{}_N-\frac{8}{3}\Theta^{M\beta}D_{i,\beta}{}^\alpha R^i]\\
[R^{a_1a_2}{}_M,P_b] &= -\frac{1}{3}g\epsilon_{MNP}\epsilon_{\alpha\beta}\Theta^{N\alpha}\delta^{[a_1}_b R^{a_2],P\beta}\\
[R^{a_1a_2a_3,\alpha},P_b] &= \frac{2}{3}g\Theta^{M\alpha}\delta^{[a_1}_b R^{a_2a_3]}{}_M\\
[R^{a_1\dots a_4,M},P_b] &= \frac{2}{3}g\epsilon_{\alpha\beta}\Theta^{M\alpha}\delta^{[a_1}_b R^{a_2a_3a_4],\beta}\\
[R^{a_1\dots a_5,\alpha}{}_M,P_b] &= \frac{1}{3}g\epsilon_{MNP}\Theta^{N\alpha}\delta^{[a_1}_b R^{a_2\dots a_5],P}\\
[R^{a_1\dots a_6,i},P_b] &= -\frac{8}{3}gD^i_{\alpha\beta}\Theta^{M\alpha}\delta^{[a_1}_b R^{a_2\dots a_6],\beta}{}_M\\
[R^{a_1\dots a_6,N}{}_M,P_b] &= -g\epsilon_{\alpha\beta}[\Theta^{N\alpha}\delta^P_M-\frac{1}{3}\Theta^{P\alpha}\delta^N_M]\delta^{[a_1}_b R^{a_2\dots a_6],\beta}{}_P\\
[R^{a_1\dots a_7,M\alpha},P_b] &= -\frac{8}{3}g\Theta^{M\beta}D_{i,\beta}{}^\alpha\delta^{[a_1}_b R^{a_2\dots a_7],i}+\frac{8}{3}g\Theta^{N\alpha}\delta^{[a_1}_b R^{a_2\dots a_7],M}{}_N\\
[R^{a_1\dots a_7,\alpha}{}_{MN},P_b] &= 0 \quad .
\end{aligned}\tag{8.20}$$

From the algebra above as well as the algebra in eq. (8.11) and using the group element of eq. (8.14) one can compute the field strengths and the gauge transformations of the fields following the general analysis of section 2. The field strength of the 1-form is

$$\begin{aligned}
F_{a_1a_2,M\alpha} = 2\Big[&\partial_{[a_1}A_{a_2],M\alpha}+\frac{g}{3}\epsilon_{MNP}\epsilon_{\alpha\beta}\Theta^{N\beta}A^P_{a_1a_2}-\frac{5g}{6}\Theta^{N\beta}A_{[a_1,N\alpha}A_{a_2],M\beta}\\
&+\frac{g}{6}\Theta^{N\beta}A_{[a_1,M\alpha}A_{a_2],N\beta}-\frac{g}{3}\Theta^N_\alpha A^\beta_{[a_1,M}A_{a_2],N\beta}\Big] \quad ,
\end{aligned}\tag{8.21}$$

the field strength of the 2-form is

$$\begin{aligned}
F^M_{a_1a_2a_3} = 3\Big[&\partial_{[a_1}A^M_{a_2a_3]}+\frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{MNP}A_{[a_1,N\alpha}\partial_{a_2}A_{a_3],P\beta}-\frac{2g}{3}\Theta^{M\alpha}A_{a_1a_2a_3,\alpha}\\
&+\frac{g}{3}\Theta^{N\alpha}A_{[a_1,N\alpha}A^M_{a_2a_3]}-\frac{g}{3}\Theta^{M\alpha}A_{[a_1,N\alpha}A^N_{a_2a_3]}\\
&+\frac{7g}{3\cdot 3!}\Theta^{P\beta}\epsilon^{MNQ}A^\alpha_{[a_1,N}A_{a_2,P\alpha}A_{a_3],Q\beta}\Big]
\end{aligned}\tag{8.22}$$

and the field strength of the 3-form is

$$F_{a_1\ldots a_4,\alpha} = 4\Big[\partial_{[a_1}A_{a_2a_3a_4],\alpha} + A_{[a_1,M\alpha}\partial_{a_2}A^M_{a_3a_4]} + \frac{1}{3!}\epsilon^{\beta\gamma}\epsilon^{MNP}A_{[a_1,M\alpha}A_{a_2,N\beta}\partial_{a_3}A_{a_4],P\gamma}$$
$$+\frac{2g}{3}\epsilon_{\alpha\beta}\Theta^{M\beta}A_{a_1\ldots a_4,M} - \frac{2g}{3}\Theta^{M\beta}A_{[a_1,M\alpha}A_{a_2a_3a_4],\beta}$$
$$+\frac{g}{6}\Theta^{N\beta}A_{[a_1,M\alpha}A_{a_2,N\beta}A^M_{a_3a_4]} - \frac{g}{6}\Theta^{M\beta}A_{[a_1,M\alpha}A_{a_2,N\beta}A^N_{a_3a_4]}$$
$$-\frac{7g}{3\cdot 4!}\Theta^{P\delta}\epsilon^{\beta\gamma}\epsilon^{MNQ}A_{[a_1,M\alpha}A_{a_2,N\beta}A_{a_3,P\gamma}A_{a_4],Q\delta}\Big] \quad . \tag{8.23}$$

These field strengths transform covariantly under the gauge transformations

$$\begin{aligned}
\delta A_{a,M\alpha} &= a_{a,M\alpha} + a_M{}^N A_{a,N\alpha} - \frac{1}{3}a_N{}^N A_{a,M\alpha} + a_i D^{i\ \beta}_{\alpha} A_{a,M\beta} \\
\delta A^M_{a_1a_2} &= a^M_{a_1a_2} - \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{MNP}A_{[a_1,N\alpha}a_{a_2],P\beta} - a_N{}^M A^N_{a_1a_2} + \frac{1}{3}a_N{}^N A^M_{a_1a_2} \\
\delta A_{a_1a_2a_3,\alpha} &= a_{a_1a_2a_3,\alpha} + A^M_{[a_1a_2}a_{a_3],M\alpha} - \frac{1}{3!}\epsilon^{MNP}\epsilon^{\beta\gamma}A_{[a_1,M\alpha}A_{a_2N\beta}a_{a_3],P\gamma} \\
&\quad + a_i D^{i\ \beta}_{\alpha} A_{a_1a_2a_3,\beta} \\
\delta A_{a_1\ldots a_4,M} &= \partial_{[a_1}\Lambda_{a_2a_3a_4],M} - \frac{1}{2}\epsilon_{MNP}A^N_{[a_1a_2}a^P_{a_3a_4]} - \epsilon^{\alpha\beta}A_{[a_1a_2a_3,\alpha}a_{a_4],\beta} \\
&\quad + \frac{1}{4!}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\epsilon^{NPQ}A_{[a_1,M\alpha}A_{a_2,N\beta}A_{a_3,P\gamma}A_{a_4],Q\delta} - \frac{1}{4}\epsilon^{\alpha\beta}A^N_{[a_1a_2}A_{a_3,M\alpha}a_{a_4],N\beta} \\
&\quad + \frac{1}{4}\epsilon^{\alpha\beta}A^N_{[a_1a_2}A_{a_3,N\alpha}a_{a_4],M\beta} + a_M{}^N A_{a_1\ldots a_4,N} - \frac{1}{3}a_N{}^N A_{a_1\ldots a_4,M} \quad ,
\end{aligned} \tag{8.24}$$

where the parameters a are given in terms of the gauge parameters Λ as

$$\begin{aligned}
a_M{}^N &= -g\Lambda_{M\alpha}\Theta^{N\alpha} \\
a^i &= \frac{8}{3}g\Theta^{M\beta}D^{i\ \beta}_{\alpha}\Lambda_{M\alpha} \\
a_{a,M\alpha} &= \partial_a\Lambda_{M\alpha} + \frac{g}{3}\epsilon_{MNP}\epsilon_{\alpha\beta}\Lambda^N_a\Theta^{P\beta} \\
a^M_{a_1a_2} &= \partial_{[a_1}\Lambda^M_{a_2]} + \frac{2}{3}g\Theta^{M\alpha}\Lambda_{a_1a_2,\alpha} \\
a_{a_1a_2a_3,\alpha} &= \partial_{[a_1}\Lambda_{a_2a_3],\alpha} - \frac{2}{3}g\epsilon_{\alpha\beta}\Theta^{M\beta}\Lambda_{a_1a_2a_3,M} \quad .
\end{aligned} \tag{8.25}$$

Using the formulae given in this paper, the reader can easily determine the field strengths and gauge transformations for the remaining fields.

8.2 Embedding tensor in the $(\overline{\mathbf{6}}, \mathbf{2})$ of $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

The deformation $(\overline{\mathbf{6}}, \mathbf{2})$, corresponding to the embedding tensor Θ^{α}_{MN} symmetric in MN, leads to the commutation relations

$$
\begin{aligned}
[R^{a,M\alpha}, P_b] &= -g\epsilon^{MNP}\Theta^{\alpha}_{NQ}\delta^{a}_{b}R^{Q}{}_{P}\\
[R^{a_1a_2}{}_{M}, P_b] &= g\epsilon_{\alpha\beta}\Theta^{\alpha}_{MN}\delta^{[a_1}_{b}R^{a_2],N\beta}\\
[R^{a_1a_2a_3,\alpha}, P_b] &= 0\\
[R^{a_1\dots a_4,M}, P_b] &= 0\\
[R^{a_1\dots a_5,\alpha}{}_{M}, P_b] &= g\Theta^{\alpha}_{MN}\delta^{[a_1}_{b}R^{a_2\dots a_5],N}\\
[R^{a_1\dots a_6,i}, P_b] &= 0\\
[R^{a_1\dots a_6,N}{}_{M}, P_b] &= g\epsilon_{\alpha\beta}\epsilon^{NPQ}\Theta^{\alpha}_{MP}\delta^{[a_1}_{b}R^{a_2\dots a_6],\beta}{}_{Q}\\
[R^{a_1\dots a_7,M\alpha}, P_b] &= 0\\
[R^{a_1\dots a_7,\alpha}{}_{MN}, P_b] &= 2g\Theta^{\alpha}_{P(M}\delta^{[a_1}_{b}R^{a_2\dots a_7],P}{}_{N)} + g\Theta^{\beta}_{MN}D_{i,\beta}{}^{\alpha}\delta^{[a_1}_{b}R^{a_2\dots a_7],i} \quad .
\end{aligned}
\tag{8.26}
$$

From the algebra above as well as the algebra in eq. (8.11) and using the group element of eq. (8.14) one can compute the field strengths and the gauge transformations of the fields corresponding to this deformation. We obtain

$$
F_{a_1a_2,M\alpha} = 2[\partial_{[a_1}A_{a_2],M\alpha} + g\epsilon_{\alpha\beta}\Theta^{\beta}_{MN}A^{N}_{a_1a_2} - \frac{g}{2}\epsilon^{NPQ}\Theta^{\beta}_{QM}A_{[a_1,N\alpha}A_{a_2],P\beta}] \tag{8.27}
$$

for the field strength of the 1-form,

$$
\begin{aligned}
F^{M}_{a_1a_2a_3} = 3\Big[&\partial_{[a_1}A^{M}_{a_2a_3]} + \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{MNP}A_{[a_1,N\alpha}\partial_{a_2}A_{a_3],P\beta} - g\epsilon^{MNQ}\Theta^{\alpha}_{PQ}A_{[a_1,N\alpha}A^{P}_{a_2a_3]}\\
&-\frac{g}{3!}\epsilon^{\alpha\beta}\epsilon^{MNT}\epsilon^{PQR}\Theta^{\gamma}_{RT}A_{[a_1,N\alpha}A_{a_2,P\beta}A_{a_3],Q\gamma}\Big]
\end{aligned}
\tag{8.28}
$$

for the field strength of the 2-form,

$$
\begin{aligned}
F_{a_1\dots a_4,\alpha} = 4\Big[&\partial_{[a_1}A_{a_2a_3a_4],\alpha} + A_{[a_1,M\alpha}\partial_{a_2}A^{M}_{a_3a_4]} + \frac{1}{3!}\epsilon^{\beta\gamma}\epsilon^{MNP}A_{[a_1,M\alpha}A_{a_2,N\beta}\partial_{a_3}A_{a_4],P\gamma}\\
&+\frac{g}{2}\epsilon_{\alpha\beta}\Theta^{\beta}_{MN}A^{M}_{[a_1a_2}A^{N}_{a_3a_4]} - \frac{g}{2}\epsilon^{MNQ}\Theta^{\beta}_{PQ}A_{[a_1,M\alpha}A_{a_2,N\beta}A^{P}_{a_3a_4]}\\
&-\frac{g}{4!}\epsilon^{MNT}\epsilon^{\beta\gamma}\epsilon^{PQR}\Theta^{\delta}_{RT}A_{[a_1,M\alpha}A_{a_2,N\beta}A_{a_3,P\gamma}A_{a_4],Q\delta}\Big]
\end{aligned}
\tag{8.29}
$$

for the field strength of the 3-form. These field strengths transform covariantly under the gauge transformations determined from eq. (8.24) once one expresses the parameters a in terms of the gauge parameters as

$$
\begin{aligned}
a_{M}{}^{N} &= g\epsilon^{NPQ}\Theta^{\alpha}_{MP}\Lambda_{Q\alpha}\\
a^{i} &= 0\\
a_{a,M\alpha} &= \partial_a\Lambda_{M\alpha} - g\epsilon_{\alpha\beta}\Theta^{\beta}_{MN}\Lambda^{N}_{a}\\
a^{M}_{a_1a_2} &= \partial_{[a_1}\Lambda^{M}_{a_2]}\\
a_{a_1a_2a_3,\alpha} &= \partial_{[a_1}\Lambda_{a_2a_3],\alpha} \quad .
\end{aligned}
\tag{8.30}
$$

Also for this deformation one can determine the field strengths and the gauge transformations of the higher rank fields using the formulae in section 2 and appendix B.

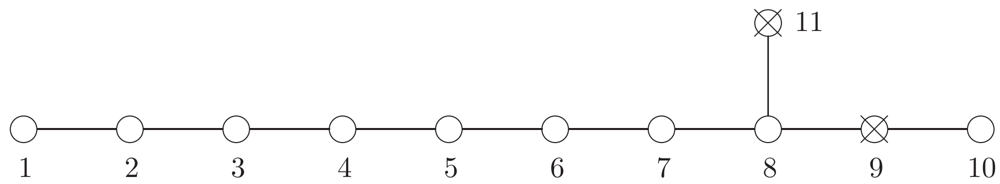

Figure 7. The E_{11} Dynkin diagram corresponding to 9-dimensional supergravity. The non-abelian part of the internal symmetry group is $\mathrm{SL}(2,\mathbb{R})$.

9 D=9

The three scalars of maximal massless nine-dimensional supergravity parametrise the manifold $\mathbb{R}^+ \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The theory also contains the metric, a doublet and a singlet of vectors, a doublet of 2-forms and a 3-form. The decomposition of E_{11} appropriate to the nine-dimensional theory is shown in figure 7. The form generators of rank less that 8 that result are associated to the fields of the supergravity theory and their duals. The E_{11} algebra also contains 8 and 9 forms, as well as generators with mixed symmetry. The generators with completely antisymmetric indices, not including the 9-forms, are

$$\begin{array}{c} R \quad R^i \quad R^a \quad R^{a,\alpha} \quad R^{a_1a_2,\alpha} \quad R^{a_1a_2a_3} \quad R^{a_1a_2a_3a_4} \quad R^{a_1\dots a_5,\alpha} \\ R^{a_1\dots a_6} \quad R^{a_1\dots a_6,\alpha} \quad R^{a_1\dots a_7} \quad R^{a_1\dots a_7,i} \quad R^{a_1\dots a_8,\alpha} \quad R^{a_1\dots a_8,i} \end{array} \quad , \tag{9.1}$$

where the $\mathrm{SL}(2,\mathbb{R})$ conventions are as in the previous section.

We now list all the non-vanishing commutators involving the operators in eq. (9.1). The scalars satisfy

$$[R^i, R^j] = f^{ij}{}_k R^k \quad , \tag{9.2}$$

while all the commutators producing the 1-forms are

$$\begin{aligned} [R, R^a] &= -R^a \qquad [R, R^{a,\alpha}] = R^{a,\alpha} \\ [R^i, R^{a,\alpha}] &= D^i_\beta{}^\alpha R^{a,\beta} \quad . \end{aligned} \tag{9.3}$$

The 2-form occurs in

$$[R^i, R^{a_1a_2,\alpha}] = D^i_\beta{}^\alpha R^{a_1a_2,\beta} \qquad [R^{a_1}, R^{a_2,\alpha}] = -R^{a_1a_2,\alpha} \quad , \tag{9.4}$$

the 3-form in

$$[R, R^{a_1a_2a_3}] = R^{a_1a_2a_3} \qquad [R^{a_1,\alpha}, R^{a_2a_3,\beta}] = \epsilon^{\alpha\beta} R^{a_1a_2a_3} \tag{9.5}$$

and the 4-form in

$$[R^{a_1a_2,\alpha}, R^{a_3a_4,\beta}] = \epsilon^{\alpha\beta} R^{a_1a_2a_3a_4} \qquad [R^{a_1}, R^{a_2a_3a_4}] = -R^{a_1a_2a_3a_4} \quad . \tag{9.6}$$

The 5-form results from the commutators

$$\begin{aligned} [R, R^{a_1\dots a_5,\alpha}] &= R^{a_1\dots a_5,\alpha} \qquad & [R^i, R^{a_1\dots a_5,\alpha}] &= D^i_\beta{}^\alpha R^{a_1\dots a_5,\beta} \\ [R^{a_1,\alpha}, R^{a_2\dots a_5}] &= R^{a_1\dots a_5,\alpha} \qquad & [R^{a_1a_2,\alpha}, R^{a_3a_4a_5}] &= R^{a_1\dots a_5,\alpha} \quad , \end{aligned} \tag{9.7}$$

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the 6-forms from the commutators

$$
\begin{aligned}
&[R, R^{a_1\dots a_6}] = 2R^{a_1\dots a_6} && [R^i, R^{a_1\dots a_6,\alpha}] = D^i_\beta{}^\alpha R^{a_1\dots a_6,\beta}\\
&[R^{a_1,\alpha}, R^{a_2\dots a_6,\beta}] = \epsilon^{\alpha\beta} R^{a_1\dots a_6} && [R^{a_1a_2a_3}, R^{a_4a_5a_6}] = R^{a_1\dots a_6}\\
&[R^{a_1}, R^{a_2\dots a_6,\alpha}] = -R^{a_1\dots a_6,\alpha} && [R^{a_1a_2,\alpha}, R^{a_3\dots a_6}] = R^{a_1\dots a_6,\alpha}
\end{aligned}
\tag{9.8}
$$

and the 7-forms from the commutators

$$
\begin{aligned}
&[R, R^{a_1\dots a_7}] = R^{a_1\dots a_7} && [R, R^{a_1\dots a_7,i}] = R^{a_1\dots a_7,i}\\
&[R^i, R^{a_1\dots a_7,j}] = f^{ij}{}_k R^{a_1\dots a_7,k} && [R^{a_1}, R^{a_2\dots a_7}] = R^{a_1\dots a_7}\\
&[R^{a_1,\alpha}, R^{a_2\dots a_7,\beta}] = D_i^{\alpha\beta} R^{a_1\dots a_7,i} + \frac{3}{4}\epsilon^{\alpha\beta} R^{a_1\dots a_7}\\
&[R^{a_1a_2,\alpha}, R^{a_3\dots a_7,\beta}] = D_i^{\alpha\beta} R^{a_1\dots a_7,i} - \frac{1}{4}\epsilon^{\alpha\beta} R^{a_1\dots a_7}\\
&[R^{a_1a_2a_3}, R^{a_4a_5a_6a_7}] = \frac{1}{2} R^{a_1\dots a_7} \quad .
\end{aligned}
\tag{9.9}
$$

Finally, the commutators giving rise to the 8-forms are

$$
\begin{aligned}
&[R, R^{a_1\dots a_8,\alpha}] = 2R^{a_1\dots a_8,\alpha} && [R^i, R^{a_1\dots a_8,\alpha}] = D^i_\beta{}^\alpha R^{a_1\dots a_8,\beta}\\
&[R^i, R^{a_1\dots a_8,j}] = f^{ij}{}_k R^{a_1\dots a_8,k} && [R^{a_1}, R^{a_2\dots a_8,i}] = -R^{a_1\dots a_8,i}\\
&[R^{a_1,\alpha}, R^{a_2\dots a_8}] = R^{a_1\dots a_8,\alpha} && [R^{a_1,\alpha}, R^{a_2\dots a_8,i}] = 3D^i_\beta{}^\alpha R^{a_1\dots a_8,\beta}\\
&[R^{a_1a_2,\alpha}, R^{a_3\dots a_8,\beta}] = D_i^{\alpha\beta} R^{a_1\dots a_8,i} && [R^{a_1a_2,\alpha}, R^{a_3\dots a_8}] = -R^{a_1\dots a_8,\alpha}\\
&[R^{a_1a_2a_3}, R^{a_3\dots a_8,\alpha}] = -\frac{1}{2} R^{a_1\dots a_8,\alpha} \quad .
\end{aligned}
\tag{9.10}
$$

One can check that all Jacobi identities are satisfied.

The algebra of eqs. (9.2)–(9.10) determines the fields strengths of all the forms of massless maximal supergravity in nine-dimensions, with the exception of the 9-forms that require the taking into account the 9-form generators. The Maurer-Cartan form that results from the group element

$$
g = e^{x\cdot P} e^{A_{a_1\dots a_8,\alpha} R^{a_1\dots a_8,\alpha}} \dots e^{A_{a_1a_2,\alpha} R^{a_1a_2,\alpha}} e^{A_{a,\alpha} R^{a,\alpha}} e^{A_a R^a} e^{\phi_i R^i} e^{\phi R}
\tag{9.11}
$$

produces indeed these field-strengths once all the spacetime indices are antisymmetrised. In this derivation of the massless theory, one imposes that all the generators in eq. (9.1) commute with momentum. We now show that, exactly as in all the other cases discussed in this paper, the field strengths of all the fields of the gauged maximal supergravities in nine dimensions result from a deformed algebra, called $\tilde{E}^{\rm local}_{11,9}$, in which the generators in eq. (9.1) have non-trivial commutation relations with the momentum operator compatibly with the Jacobi identities.

As usual, from the E_{11} perspective the commutator of the 1-form generators with momentum give rise to the scalar generators contracted with the embedding tensor. Therefore, in this nine-dimensional case the embedding tensor in contained in the SL$(2,\mathbb{R})$ tensor product

$$
(\mathbf{1}\oplus\mathbf{2})\otimes(\mathbf{1}\oplus\mathbf{3}) = \mathbf{4}\oplus\mathbf{3}\oplus\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{1} \quad .
\tag{9.12}
$$

The singlet Θ would correspond to the commutator

$$[R^a, P_b] = -g\Theta\delta^a_b R \quad , \tag{9.13}$$

which is ruled out because of the Jacobi identity involving R^a, R^b and P_c. Similarly, the quadruplet $\Theta^{\alpha\beta\gamma}$ would lead to the commutator

$$[R^{a,\alpha}, P_b] = -g\Theta^{\alpha\beta\gamma}D_{i,\beta\gamma}\delta^a_b R^i \quad , \tag{9.14}$$

which is ruled out because of the Jacobi identity involving $R^{a,\alpha}$, $R^{b,\beta}$ and P_c. The two doublet deformations lead to the commutator

$$[R^{a,\alpha}, P_b] = -g\delta^a_b[a\Theta^\alpha R + b\Theta^\beta D_{i,\beta}{}^\alpha R^i] \quad , \tag{9.15}$$

and the Jacobi identities impose the condition

$$b = -4a \quad . \tag{9.16}$$

This leads to one possible doublet deformation, and we fix the parameter a to 1. To summarise, the only representations of the embedding tensor which are not ruled out are the triplet and one of the two doublets. We now show that both these embedding tensors lead to a consistent algebra. In [28] all the possible gauged maximal supergravities in nine dimensions were constructed. They indeed correspond to an embedding tensor in either the triplet or the doublet of SL(2, $\mathbb{R}$).

9.1 Embedding tensor in the 3 of SL(2, $\mathbb{R}$)

We first consider the deformation in the triplet, which corresponds to the embedding tensor Θ_i. This case as been considered in [17] up to the 5-forms. In that paper the deformation parameter was denoted with m_i, and our conventions here are such that $m_i = -g\Theta_i$. Deforming from the commutator of R^a with momentum, all the other commutators are determined by requiring that the Jacobi identities close. The final result is

$$\begin{aligned}
[R^a, P_b] &= -g\delta^a_b\Theta_i R^i \\
[R^{a,\alpha}, P_b] &= 0 \\
[R^{a_1a_2,\alpha}, P_b] &= g\Theta_i D^i_\beta{}^\alpha \delta^{[a_1}_b R^{a_2],\beta} \\
[R^{a_1a_2a_3}, P_b] &= 0 \\
[R^{a_1\dots a_4}, P_b] &= 0 \\
[R^{a_1\dots a_5,\alpha}, P_b] &= 0 \\
[R^{a_1\dots a_6}, P_b] &= 0 \\
[R^{a_1\dots a_6,\alpha}, P_b] &= g\Theta_i D^i_\beta{}^\alpha \delta^{[a_1}_b R^{a_2\dots a_6],\beta} \\
[R^{a_1\dots a_7}, P_b] &= 0 \\
[R^{a_1\dots a_7,i}, P_b] &= -g\Theta^i\delta^{[a_1}_b R^{a_2\dots a_7]} \\
[R^{a_1\dots a_8,\alpha}, P_b] &= 0 \\
[R^{a_1\dots a_8,i}, P_b] &= -g\Theta^i\delta^{[a_1}_b R^{a_2\dots a_8]} - gf^{ij}{}_k\Theta_j\delta^{[a_1}_b R^{a_2\dots a_8],k} \quad .
\end{aligned} \tag{9.17}$$

From these commutation relations and the massless algebra of eqs. (9.2)–(9.10), using the group element in eq. (9.11), one determines the field strengths and the gauge transformations on the fields. The field strengths of the 1-forms are

$$\begin{aligned} F_{a_1a_2} &= 2\partial_{[a_1}A_{a_2]} \\ F_{a_1a_2,\alpha} &= 2[\partial_{[a_1}A_{a_2],\alpha} - g\Theta_i D^{i\ \beta}_{\alpha} A_{a_1a_2,\beta}] \quad , \end{aligned} \tag{9.18}$$

the field-strength of the 2-form is

$$F_{a_1a_2a_3,\alpha} = 3[\partial_{[a_1}A_{a_2a_3],\alpha} - A_{[a_1}\partial_{a_2}A_{a_3],\alpha} + g\Theta_i D^{i\ \beta}_{\alpha} A_{[a_1}A_{a_2a_3],\beta}] \quad , \tag{9.19}$$

the field strength of the 3-form is

$$F_{a_1\dots a_4} = 4[\partial_{[a_1}A_{a_2\dots a_4]} + \epsilon^{\alpha\beta}A_{[a_1}\partial_{a_2}A_{a_3a_4],\beta} + \frac{g}{2}\Theta^i D_i^{\alpha\beta} A_{[a_1a_2,\alpha}A_{a_3a_4],\beta}] \quad , \tag{9.20}$$

the field strength of the 4-form is

$$\begin{aligned} F_{a_1\dots a_5} = 5\Big[&\partial_{[a_1}A_{a_2\dots a_5]} - A_{[a_1}\partial_{a_2}A_{a_3a_4a_5]} - \frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1a_2,\alpha}\partial_{a_3}A_{a_4a_5],\beta} \\ &-\epsilon^{\alpha\beta}A_{[a_1}A_{a_2,\alpha}\partial_{a_3}A_{a_4a_5],\beta} - \frac{g}{2}\Theta^i D_i^{\alpha\beta}A_{[a_1}A_{a_2a_3,\alpha}A_{a_4a_5],\beta}\Big] \end{aligned} \tag{9.21}$$

and the field strength of the 5-from is

$$\begin{aligned} F_{a_1\dots a_6,\alpha} = 6\Big[&\partial_{[a_1}A_{a_2\dots a_6],\alpha} + A_{[a_1,\alpha}\partial_{a_2}A_{a_3\dots a_6]} - A_{[a_1a_2,\alpha}\partial_{a_3}A_{a_4a_5a_6]} \\ &-\frac{1}{2}\epsilon^{\beta\gamma}A_{[a_1,\alpha}A_{a_2a_3,\beta}\partial_{a_4}A_{a_5a_6],\gamma} - \frac{g}{3!}\Theta^i D_i^{\beta\gamma}A_{[a_1a_2,\alpha}A_{a_3a_4,\beta}A_{a_5a_6],\gamma} \\ &-g\Theta_i D^{i\ \beta}_{\alpha}A_{a_1\dots a_6,\beta}\Big] \quad . \end{aligned} \tag{9.22}$$

These field strengths transform covariantly under the gauge transformations

$$\begin{aligned} \delta A_a &= a_a - aA_a \\ \delta A_{a,\alpha} &= a_{a,\alpha} - aA_{a,\alpha} + a_i D^{i\ \beta}_{\alpha} A_{a,\beta} \\ \delta A_{a_1a_2,\alpha} &= a_{a_1a_2,\alpha} + A_{[a_1,\alpha}a_{a_2]} + a_i D^{i\ \beta}_{\alpha} A_{a_1a_2,\beta} \\ \delta A_{a_1a_2a_3} &= a_{a_1a_2a_3} - \epsilon^{\alpha\beta}A_{[a_1a_2,\alpha}a_{a_3],\beta} + \frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1,\alpha}A_{a_2,\beta}a_{a_3]} + aA_{a_1a_2a_3} \\ \delta A_{a_1\dots a_4} &= a_{a_1\dots a_4} - \frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1a_2,\alpha}a_{a_3a_4],\beta} + A_{[a_1a_2a_3}a_{a_4]} + \frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1a_2,\alpha}A_{a_3,\beta}a_{a_4]} \\ \delta A_{a_1\dots a_5,\alpha} &= a_{a_1\dots a_5,\alpha} + A_{[a_1a_2a_3}a_{a_4a_5],\alpha} + A_{[a_1\dots a_4}a_{a_5],\alpha} - \frac{1}{2}\epsilon^{\beta\gamma}A_{[a_1a_2,\alpha}A_{a_3a_4,\beta}a_{a_5],\gamma} \\ &\quad +\frac{1}{2}\epsilon^{\beta\gamma}A_{[a_1a_2,\alpha}A_{a_3,\beta}A_{a_4,\gamma}a_{a_5]} + aA_{a_1\dots a_5,\alpha} + a_i D^{i\ \beta}_{\alpha}A_{a_1\dots a_5,\beta} \\ \delta A_{a_1\dots a_6} &= \partial_{[a_1}\Lambda_{a_2\dots a_6]} - \frac{1}{2}A_{[a_1a_2a_3}a_{a_4a_5a_6]} + \epsilon^{\alpha\beta}A_{[a_1\dots a_5,\alpha}a_{a_6],\beta} \\ &\quad -\frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1a_2a_3}A_{a_4a_5,\alpha}a_{a_6],\beta} + \frac{1}{4}A_{[a_1a_2a_3}A_{a_4,\alpha}A_{a_5,\beta}a_{a_6]} + 2aA_{a_1\dots a_6} \\ \delta A_{a_1\dots a_6,\alpha} &= \partial_{[a_1}\Lambda_{a_2\dots a_6],\alpha} + A_{[a_1\dots a_4}a_{a_5a_6],\alpha} - \frac{1}{3!}\epsilon^{\beta\gamma}A_{[a_1a_3,\alpha}A_{a_3a_4,\beta}a_{a_5a_6],\gamma} + A_{[a_1\dots a_5,\alpha}a_{a_6]} \\ &\quad -\frac{1}{2}\epsilon^{\beta\gamma}A_{[a_1a_2,\alpha}A_{a_3a_4,\beta}A_{a_5,\gamma}a_{a_6]} + a_i D^{i\ \beta}_{\alpha}A_{a_1\dots a_6,\beta} \quad , \end{aligned} \tag{9.23}$$

where the parameters a are given in terms of the gauge parameters Λ as

$$
\begin{aligned}
a &= 0 \\
a^i &= -g\Lambda\Theta^i \\
a_a &= \partial_a \Lambda \\
a_{a,\alpha} &= \partial_a \Lambda_\alpha + g\Theta_i D^{i\ \beta}_{\alpha} \Lambda_{a,\beta} \\
a_{a_1 a_2,\alpha} &= \partial_{[a_1} \Lambda_{a_2],\alpha} \\
a_{a_1 a_2 a_3} &= \partial_{[a_1} \Lambda_{a_2 a_3]} \\
a_{a_1 \dots a_4} &= \partial_{[a_1} \Lambda_{a_2 a_3 a_4]} \\
a_{a_1 \dots a_5,\alpha} &= \partial_{[a_1} \Lambda_{a_2 \dots a_5],\alpha} + g\Theta_i D^{i\ \beta}_{\alpha} \Lambda_{a_1 \dots a_5,\beta} \quad .
\end{aligned} \tag{9.24}
$$

The reader can easily evaluate the remaining field strengths and gauge transformations.

9.2 Embedding tensor in the 2 of $\mathrm{SL}(2,\mathbb{R})$

We now consider the doublet deformation, corresponding to the embedding tensor Θ^α. We start from the commutator between the 1-form and momentum as in eq. (9.15) with the parameters as in eq. (9.16) with $a = 1$. Imposing the closure of the Jacobi identities gives

$$
\begin{aligned}
[R^a, P_b] &= 0 \\
[R^{a,\alpha}, P_b] &= -g\delta^a_b [\Theta^\alpha R - 4\Theta^\beta D_{i,\beta}{}^\alpha R^i] \\
[R^{a_1 a_2,\alpha}, P_b] &= g\Theta^\alpha \delta^{[a_1}_b R^{a_2]} \\
[R^{a_1 a_2 a_3}, P_b] &= -g\epsilon_{\alpha\beta}\Theta^\alpha \delta^{[a_1}_b R^{a_2 a_3],\beta} \\
[R^{a_1 \dots a_4}, P_b] &= 0 \\
[R^{a_1 \dots a_5,\alpha}, P_b] &= 0 \\
[R^{a_1 \dots a_6}, P_b] &= -2g\epsilon_{\alpha\beta}\Theta^\alpha \delta^{[a_1}_b R^{a_2 \dots a_6],\beta} \\
[R^{a_1 \dots a_6,\alpha}, P_b] &= 0 \\
[R^{a_1 \dots a_7}, P_b] &= -2g\epsilon_{\alpha\beta}\Theta^\alpha \delta^{[a_1}_b R^{a_2 \dots a_7],\beta} \\
[R^{a_1 \dots a_7,i}, P_b] &= -2g\Theta^\alpha D^i_{\alpha\beta} \delta^{[a_1}_b R^{a_2 \dots a_7],\beta} \\
[R^{a_1 \dots a_8,\alpha}, P_b] &= \frac{1}{2} g\Theta^\alpha \delta^{[a_1}_b R^{a_2 \dots a_8]} - 2gD_{i,\beta}{}^\alpha \Theta^\beta \delta^{[a_1}_b R^{a_2 \dots a_8],i} \\
[R^{a_1 \dots a_8,i}, P_b] &= 0 \quad .
\end{aligned} \tag{9.25}
$$

From these commutation relations and the massless algebra of eqs. (9.2)–(9.10), using the group element in eq. (9.11), one determines the field strengths and the gauge transformations on the fields. The field strengths of the 1-forms are

$$
\begin{aligned}
F_{a_1 a_2} &= 2[\partial_{[a_1} A_{a_2]} - g\Theta^\alpha A_{a_1 a_2,\alpha} + g\Theta^\alpha A_{[a_1} A_{a_2],\alpha}] \\
F_{a_1 a_2,\alpha} &= 2\left[\partial_{[a_1} A_{a_2],\alpha} - g\Theta^\beta A_{[a_1,\alpha} A_{a_2],\beta} - \frac{g}{2}\Theta_\alpha A^\beta_{[a_1} A_{a_2],\beta}\right] \quad ,
\end{aligned} \tag{9.26}
$$

the field-strength of the 2-form is

$$F_{a_1a_2a_3,\alpha} = 3\Big[\partial_{[a_1}A_{a_2a_3],\alpha} - A_{[a_1}\partial_{a_2}A_{a_3],\alpha} - g\epsilon_{\alpha\beta}\Theta^\beta A_{a_1a_2a_3} + g\Theta^\beta A_{[a_1,\alpha}A_{a_2a_3],\beta} \\ +g\Theta^\beta A_{[a_1}A_{a_2,\alpha}A_{a_3],\beta} + \frac{g}{2}\Theta_\alpha A_{[a_1}A^\beta_{a_2}A_{a_3],\beta}\Big] \quad , \tag{9.27}$$

the field strength of the 3-form is

$$F_{a_1\dots a_4} = 4\Big[\partial_{[a_1}A_{a_2\dots a_4]} + \epsilon^{\alpha\beta}A_{[a_1}\partial_{a_2}A_{a_3a_4],\beta} + g\Theta^\alpha A_{[a_1,\alpha}A_{a_2a_3a_4]} \\ -\frac{g}{2}\Theta^\alpha A^\beta_{[a_1}A_{a_2,\beta}A_{a_3a_4],\alpha}\Big] \quad , \tag{9.28}$$

the field strength of the 4-form is

$$F_{a_1\dots a_5} = 5\Big[\partial_{[a_1}A_{a_2\dots a_5]} - A_{[a_1}\partial_{a_2}A_{a_3a_4a_5]} - \frac{1}{2}\epsilon^{\alpha\beta}A_{[a_1a_2,\alpha}\partial_{a_3}A_{a_4a_5],\beta} \\ -\epsilon^{\alpha\beta}A_{[a_1}A_{a_2,\alpha}\partial_{a_3}A_{a_4a_5],\beta} - g\Theta^\alpha A_{[a_1a_2,\alpha}A_{a_3a_4a_5]} \\ -g\Theta^\alpha A_{[a_1}A_{a_2,\alpha}A_{a_3a_4a_5]} - \frac{g}{2}\epsilon^{\alpha\beta}\Theta^\gamma A_{[a_1}A_{a_2,\alpha}A_{a_3,\beta}A_{a_4a_5],\gamma}\Big] \tag{9.29}$$

and the field strength of the 5-from is

$$F_{a_1\dots a_6,\alpha} = 6\Big[\partial_{[a_1}A_{a_2\dots a_6],\alpha} + A_{[a_1,\alpha}\partial_{a_2}A_{a_3\dots a_6]} - A_{[a_1a_2,\alpha}\partial_{a_3}A_{a_4a_5a_6]} \\ -\frac{1}{2}\epsilon^{\beta\gamma}A_{[a_1,\alpha}A_{a_2a_3,\beta}\partial_{a_4}A_{a_5a_6],\gamma} - 2g\epsilon_{\alpha\beta}\Theta^\beta A_{a_1\dots a_6} \\ -g\Theta^\beta A_{[a_1,\alpha}A_{a_2a_3,\beta}A_{a_4a_5a_6]}\Big] \quad . \tag{9.30}$$

These field strengths transform covariantly under the gauge transformations in eq. (9.23), with the parameters a given in terms of the gauge parameters Λ as

$$\begin{aligned} a &= -g\Theta^\alpha\Lambda_\alpha \\ a^i &= 4g\Theta^\beta D^{i\ \alpha}_\beta\Lambda_\alpha \\ a_a &= \partial_a\Lambda + g\Theta^\alpha\Lambda_{a,\alpha} \\ a_{a,\alpha} &= \partial_a\Lambda_\alpha \\ a_{a_1a_2,\alpha} &= \partial_{[a_1}\Lambda_{a_2],\alpha} + g\epsilon_{\alpha\beta}\Theta^\beta\Lambda_{a_1a_2} \\ a_{a_1a_2a_3} &= \partial_{[a_1}\Lambda_{a_2a_3]} \\ a_{a_1\dots a_4} &= \partial_{[a_1}\Lambda_{a_2a_3a_4]} \\ a_{a_1\dots a_5,\alpha} &= \partial_{[a_1}\Lambda_{a_2\dots a_5],\alpha} + 2g\epsilon_{\alpha\beta}\Theta^\beta\Lambda_{a_1\dots a_5} \quad . \end{aligned} \tag{9.31}$$

The reader can easily evaluate the remaining field strengths and gauge transformations corresponding to this deformation.

D	G	1-forms	2-forms	3-forms	4-forms	5-forms	6-forms	7-forms	8-forms	9-forms	10-forms
10A	$\mathbb{R}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$		$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$ $\mathbf{1}$
10B	$SL(2,\mathbb{R})$		$\mathbf{2}$		$\mathbf{1}$		$\mathbf{2}$		$\mathbf{3}$		$\mathbf{4}$ $\mathbf{2}$
9	$SL(2,\mathbb{R})\times\mathbb{R}^+$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{2}$	$\mathbf{4}$ $\mathbf{2}$ $\mathbf{2}$	
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{2})$ $(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{15},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$		
7	$SL(5,\mathbb{R})$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{40}}$ $\overline{\mathbf{15}}$	$\mathbf{70}$ $\mathbf{45}$ $\mathbf{5}$			
6	$SO(5,5)$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\mathbf{144}$	$\mathbf{320}$ $\overline{\mathbf{126}}$ $\mathbf{10}$				
5	$E_{6(+6)}$	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\mathbf{351}$	$\overline{\mathbf{1728}}$ $\overline{\mathbf{27}}$					
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645}$ $\mathbf{133}$						
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{3875}$ $\mathbf{248}$							

Table 1. Table giving the representations of the symmetry group G of all the forms fields of maximal supergravities in any dimension [13]. The 3-forms in three dimensions were determined in [14]. It is important to observe that these are the representations of the fields, which are the contragredient of the representations of the corresponding generators, which have been considered in this paper.

10 Form field equations and duality conditions

In this section we write down the equations of motion for the form fields taking into account that we have fields and their duals. Such equations have been studied on an ad-hoc basis previously beginning with [29]. However, our discussions will be in the context of E_{11} and in particular the representations and hierarchy of form fields it predicts [13] and as is given in the table of [13] that is table 1 of this paper.

If we assume that the form field equations are first order in space-time derivatives they can only be duality relations between the field strengths obtained in this paper. Let us first consider gauge fields whose field strengths have a rank that is not half that of the dimensions of space-time, that is those that do not obey some kind of generalised self duality condition. Examining the table 1 of the representations of G of the form fields we find that for every gauge field of rank n for $n < \frac{1}{2}D$, with a field strength F_{n+1} of rank $n+1$, that belongs to a representation $\mathbf{R_n}$ there is a dual gauge field of rank $D-n-2$ with a field strength F_{D-n-1} of rank $D-n-1$ which is in that representation $\mathbf{R_{D-n-2}}$ which turns out to be the conjugate representation, i.e. $\mathbf{R_{D-n-2}} = \overline{\mathbf{R}}_\mathbf{n}$. The field strengths that occur in the Cartan forms transform under G with a non-linear action that is only a transformation under the Cartan involution invariant subgroup $I(G)$ rather than the above mentioned linear representations of G. This is due to the scalar factors mentioned above which convert the linear representation into the non-linear representation in the well know manner. Thus demanding invariant field equations reduces to finding those invariant under only $I(G)$ transformations. Examining all such gauge fields for $D \leq 7$, we find that under the decomposition of their representations of G from G to $I(G)$ we find one irreducible real representation of $I(G)$. Hence the gauge fields and their duals belong to the same representation of $I(G)$. For example in seven dimensions the two forms belong to the **5** of $\mathrm{SL}(5,\mathbb{R})$ while their dual gauge fields, the three forms, belong to the $\overline{\mathbf{5}}$ of $\mathrm{SL}(5,\mathbb{R})$. The Cartan involution invariant subgroup is $I(\mathrm{SL}(5,\mathbb{R})) = \mathrm{SO}(5)$ and these two gauge fields both belong to the real **5** representation of this group.

The field equations for all such gauge fields can only be of the form

$$F_{n+1} = \star F_{D-n-1} \quad , \tag{10.1}$$

where $\star$ is the space-time dual, since they each belong to the same irreducible representation of $I(G)$. For dimensions $D \geq 8$ the gauge fields belong to representations of G that decompose into at most two distinct irreducible real representations of $I(G)$ and their dual gauge fields belong to precisely the same representations of $I(G)$. Then the duality condition consists of as many equations as there are representations of $I(G)$, which are of the form of eq. (10.1) and they relate the gauge field and its dual in the same representation of $I(G)$.

The scalars are a non-linear realisation of G and obey duality relations with the rank $D-2$ forms which are in the adjoint representation. Under the decomposition from G to $I(G)$ the adjoint representation of the latter breaks into the adjoint of $I(G)$ and the "coset" part. Only the latter enters into the duality condition with the coset part of the Cartan form formed from the scalars. The scalar equation results from the curl of these duality relations. Such curl reproduces the field strengths of the $D-1$ form fields, which are dual to the embedding tensor, and this gives rise to the scalar potential. In general there is more than one gauge covariant quantity that one can construct contracting the scalars with the embedding tensor, and the method we have presented in this paper of determining all the gauge covariant quantities of the theory does not determine their relative coefficient, and therefore does not determine the exact form of the scalar potential.

For odd dimensions space-times there are clearly no generalised self duality conditions. However, the cases when $D = 4m$ and $D = 4m+2$, for integer m are different due to the fact that $\star\star = -1$ and $\star\star = +1$ respectively when acting on a $\frac{D}{2}$-form. Let us begin with the latter case, that is dimensions ten and six. As is well known in ten dimensions we have a four form gauge field that is a singlet of $\mathrm{SL}(2,\mathbb{R})$ and obeys a self duality condition of the form $F_5 = \star F_5$. In six dimensions that two forms belong to the **10** of SO(5,5) which decomposes into the reducible representation $(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})$ of $\mathrm{SO}(5)\otimes\mathrm{SO}(5) = I(\mathrm{SO}(5,5))$. The duality condition which is invariant under the $\mathrm{SO}(5) \otimes SO(5)$ transformations of the field strength can only be of the form

$$\begin{pmatrix} F_3 \\ F_3' \end{pmatrix} = \star \begin{pmatrix} F_3 \\ -F_3' \end{pmatrix} \quad , \tag{10.2}$$

where F_3 and F_3' belong to the $(5,1)$ and $(1,5)$ representations of $\mathrm{SO}(5)\otimes\mathrm{SO}(5)$ respectively. The minus sign is required as there must be the same number of self-dual and anti-self-dual forms as the resulting theory describes 5 tensors that do not satisfy self-duality conditions.

Let us now consider the case of $D = 4m + 2$ that is dimensions eight and four. In this case the forms belong to an irreducible representation of G that breaks into two irreducible representations of $I(G)$ which are related by complex conjugation. In eight dimensions the three forms belong to the $(\mathbf{1}, \mathbf{2})$ representation of $\mathrm{SL}(3,\mathbb{R}) \otimes \mathrm{SL}(2,\mathbb{R})$ which breaks into $(\mathbf{1}, \mathbf{1}^+)$ and $(\mathbf{1}, \mathbf{1}^-)$ representations of $I(\mathrm{SL}(3,\mathbb{R}) \otimes \mathrm{SL}(2,\mathbb{R})) = \mathrm{SO}(3) \otimes \mathrm{SO}(2)$. As such the unique invariant field equation is of the form

$$\begin{pmatrix} F_4 \\ F_4^* \end{pmatrix} = i \star \begin{pmatrix} F_4 \\ -F_4^* \end{pmatrix} \quad , \tag{10.3}$$

where F_4 and F_4^* belong to the $(1, 1^+)$ and $(1, 1^-)$ representations of $\mathrm{SO}(3)\otimes\mathrm{SO}(2)$. The i found in this equation is due to the fact that $\star\star = -1$ in this dimension and the minus sign then results from the consistency with respect to complex conjugation. In four dimensions the one forms belong to the $\mathbf{56}$ dimensional representation of E_7 which decompose into the $\mathbf{28} \oplus \overline{\mathbf{28}}$ of representations $I(E_7) = \mathrm{SU}(8)$. The self duality condition can only be of the form

$$\begin{pmatrix} F_2 \\ F_2^* \end{pmatrix} = i \star \begin{pmatrix} F_2 \\ -F_2^* \end{pmatrix} \quad , \tag{10.4}$$

where F_2 and F_2^* are the $\mathbf{28} \oplus \overline{\mathbf{28}}$ representations of $\mathrm{SU}(8)$.

These equations of motion are the correct equations although we have not derived these duality relations as following from E_{11} in this paper. This remains a future project. There is a certain freedom to rescale the form fields by constants which is reflected that these duality relations can have constants that are not explicitly shown above. These constants are fixed once one also writes down the field equation for gravity as this involves the stress tensor. It is impressive to see the way the representations of the form fields, dictated by E_{11}, cooperate with the demands that the form field equations be duality conditions.

The duality relations discussed in this section have a crucial role in determining the closure of the supersymmetry algebra. Indeed, these relations are first order in derivatives, and the closure of the supersymmetry algebra on fields and dual fields, as well as on $D-1$ and D forms, only occurs provided that they are satisfied. In [30] and [31] it was shown that the supersymmetry algebra of IIB and IIA respectively close on all the fields and dual fields provided that the duality relations are satisfied. The supersymmetry algebra also fixes the $D-1$ and D forms that one can include, the result being exactly in agreement with the predictions of E_{11} [12] (subsequently, it was shown in [32] that also the detailed coefficients of the gauge algebra of the IIB theory are reproduced by E_{11}). In particular the IIA algebra describes both the massive and massless field equations, as the field strength of the 9-form

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can be set equal to the Romans cosmological constant [33] or to zero respectively. More recently, the closure of the supersymmetry algebra on higher rank forms was shown for the case of gauged maximal supergravities in five dimensions [16] and in three dimensions [34].

11 Conclusions

It was previously found [13, 14] that the maximal gauged supergravity theories were classified by E_{11}. In particular the forms of rank $D-1$ in the D dimensional supergravity theory, which lead to a cosmological constants, are in the contragredient representation of the internal symmetry group of the tensor which was known to label all such theories. Although this discovery was kinematical in nature it demonstrated that E_{11} provided, for the first time, a unifying scheme within which to consider all such theories. In this paper we show that E_{11}, by the steps described in this paper, leads to all the field strengths of the maximal gauged supergravity theories. The embedding tensor arises as the tensor that uniquely determines the deformation of the E_{11} algebra from which the gauged supergravities arise as non-linear realisations and we show that it is in the same representation as the $D-1$ form generators. We have analysed each dimension from three to nine, and these results, together with the ten-dimensional deformation corresponding to the Romans theory analysed in [17], give the field strengths of all possible massive maximal supergravities in any dimension.

If one assumes, as is the case, that the dynamics of the form fields are first order in space-time derivatives then they must be given by duality relations on the field strengths calculated in this paper. As a result, for all the fields apart from the scalars, the dynamics of the bosonic sector is then determined up to a few constants that multiply the field strengths. In the absence of the gravity equation that contains the stress tensor one can fix these constants by field redefinitions, hence in this sense the dynamics is determined in the absence of gravity (in the case of the scalar equation in general there is more than one gauge covariant quantity that one can construct contracting the scalars with the embedding tensor, and the procedure presented in this paper does not determine their relative coefficient, and therefore does not determine the exact form of the scalar potential). Thus most results on the maximal gauged supergravity theories, including those that have been derived over many years, can be found in a very quick, efficient and unified manner from E_{11}.

In the E_{11} formulation of the maximal gauged supergravities theories the field content in a given dimension is the same although the actual physical degree of freedom in any given gauged supergravity theory may differ. In particular the number of $D-1$ forms is the same and so a given maximal gauged supergravity theory has a knowledge of all the other possible gauged supergravity theories in the same dimension. This is analogous to having various different theories and then discovering that there is a potential from which they can all be derived as different minima.

In this paper we have used a deformation of an algebra $\tilde{E}_{11}$ containing the E_{11} algebra, the usual space-time translation and the Ogievetsky generators. However, in reference [17] a detailed study of the nine dimensional gauged supergravities was carried out and it was found that these theories arose from the full $E_{11,10B}^{\text{local}}$ including the parts associated with ten dimensions. However, these gauged supergravities could be constructed from only a subalgebra of $\tilde{E}_{11,9}^{\text{local}}$ which appeared to be a deformed $E_{11,9}^{\text{local}}$ algebra as a result of the complicated field redefinitions of the generators and the generators that were dropped as they played no role in the dynamics. As such in these theories one is dealing with a subalgebra of $\tilde{E}_{11,9}^{\text{local}}$ which only appears as a deformation as a consequence of the way the calculation is carried out. It would be of interest to see if this is a general phenomenon.

As discussed in the introduction, the original gauged supergravities were derived by adding a deformations to the massless theory and using supersymmetry to find the complete the theory. In such an approach one did not use fields that were in representations of the internal symmetry group G. Later gauged maximal supergravities were constructed using fields that were representations of G, but the theory also contained an embedding tensor that labelled the theories and broke the internal symmetry group G. In this way of proceeding the fields always occurred together with this tensor in just such a way that the full G representations of the fields was not present into the equations of motion. However, in the last few years the bosonic sector of certain gauged supergravities have been constructed [35] by taking the physical degrees of freedom to be described by fields that are representations of the internal symmetry and demanding that these unconstrained fields carry a gauge algebra extending the gauging of part of the local internal symmetry. In carrying out this programme these authors have found a hierarchy of fields of increasing rank [34]. However, these are just those found previously in the E_{11} approach [13]. It is obvious that this procedure is just a bottom up way of discovering the form sector of E_{11} and it is not necessary to speculate about the mysterious degrees of freedom of M theory that such a process may have uncovered.

While there can be no doubt of the calculational efficiency of the approach of this paper it leaves open a number of more conceptual questions. For example, what mathematical object do the Ogievetsky generators belong to. Also even though one uses only the positive and zero level part of the E_{11} algebra in the deformed algebra, this algebra is only defined from the full E_{11} algebra. As a result many properties of, and deductions from, the full E_{11} algebra are imported into the calculation. This would at best seem unnatural. In a previous paper it was proposed to derive the gauged supergravity theories, and explicitly the five dimensional gauged supergravities, from a non-linear realisation of $E_{11} \otimes_s l_1$ where l_1 is the fundamental representation associated to the first node of E_{11} and leads to a generalised space-time. This approach has the advantage that it is conceptually well defined from the mathematical viewpoint, but it is less clear how some of the physical aspects of the gauged supergravity theories emerge in a natural way. These include how the local gauge transformations arise and how the slice of generalised space-time that is active arises from the full generalised space-time. Thus one has a dilemma how to include space-time, local gauge transformations within E_{11}. In this context we might mention the interesting work of reference [36] that concerns the role of diffeomorphism symmetries in the context of the non-linear realisation relevant to supergravity theories. We hope to report elsewhere on progress in this area.

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A Group theory conventions and projectors

In this appendix we first review some of the group-theoretic techniques that have been used in this paper, and we then discuss the E_{11} derivation of various representation projectors, fucusing in particular on the cases of E_7 and E_6 which have been discussed in sections 4 and 5. These projectors arise in E_{11} as conditions on the structure constants that contract the $D-1$ form generators, and the fact that the consistency of the algebra imposes that the embedding tensor must satisfy the same projection conditions proves that the embedding tensor and the $D-1$-form generators must belong to the same representation. At the end of this appendix we then show that these projectors are precisely the ones that result from a purely group theoretic analysis based on the representations of the internal symmetry group.

The Cartan-Killing metric $\kappa^{\alpha\beta}$ is defined as

$$C_{\rm Adj}\kappa^{\alpha\beta} = f^{\alpha\gamma}{}_{\epsilon} f^{\beta\epsilon}{}_{\gamma} \quad , \tag{A.1}$$

where $C_{\rm Adj}$ is the quadratic Casimir in the adjoint representation. Denoting with $\mathbf{D_\Lambda}$ the fundamental representation, one then defines the quadratic Casimir in the fundamental representation C_Λ from the relation

$$C_\Lambda \delta_M^N = \kappa_{\alpha\beta} D^\alpha_M{}^P D^\beta_P{}^N \quad . \tag{A.2}$$

When not otherwise specified in the paper, we use to raise and lower indices in the adjoint representation the metric

$$g^{\alpha\beta} = {\rm Tr}(D^\alpha D^\beta) = D^\alpha_M{}^N D^\beta_N{}^M \quad , \tag{A.3}$$

where the trace is in the fundamental, i.e. lowest dimensional, representation. This metric differs from the Cartan-Killing metric of eq. (A.1) by a constant, and indeed from eq. (A.2) one finds

$$\kappa^{\alpha\beta} = \frac{d}{C_\Lambda d_\Lambda} g^{\alpha\beta} \quad , \tag{A.4}$$

where d is the dimension of the adjoint and d_Λ is the dimension of the fundamental representation. Substituting the inverse of eq. (A.4) in eq. (A.2) one also derives

$$g_{\alpha\beta} D^\alpha_M{}^P D^\beta_P{}^N = \frac{d}{d_\Lambda}\delta_M^N \quad , \tag{A.5}$$

while substituting it in eq. (A.1) one gets

$$f^{\alpha\beta\gamma} f_{\alpha\beta\delta} = -\frac{d}{d_\Lambda}\frac{C_{\rm Adj}}{C_\Lambda}\delta_\delta^\gamma \quad , \tag{A.6}$$

as follows from raising the indices using the metric in eq. (A.3). The ratio $\frac{C_{\rm Adj}}{C_\Lambda}$ is given by the relation

$$\frac{C_{\rm Adj}}{C_\Lambda} = \frac{d_\Lambda}{d}\frac{g^\vee}{\tilde{I}_\Lambda} \quad , \tag{A.7}$$

where $g^\vee$ is the dual Coxeter number and $\tilde{I}_\Lambda$ is the Dynkin index of the fundamental representation.

In this paper we have shown that the deformed E_{11} algebras resulting from suitably modifying the commutation relations of the E_{11} generators with momentum are entirely classified by the tensor $\Theta_\alpha^{M_1}$ arising in eq. (2.13), where M_1 denotes the representation of the 1-form E_{11} generator R^{a_1,M_1} in a given dimension. As we have shown, this tensor is identified with the embedding tensor. Denoting with $\mathbf{R_1}$ this representation, and with $\mathbf{R_0}$ the adjoint, the embedding tensor is contained in the tensor product $\mathbf{R_1} \otimes \mathbf{R_0}$. Since the $D-2$-form generators also belong to the adjoint, that is $\mathbf{R_{D-2}} = \mathbf{R_0}$, this tensor product is the same as the tensor product $\mathbf{R_1} \otimes \mathbf{R_{D-2}}$, which occurs in the commutator between the 1-form and the $D-2$-form. This commutator gives rise to the $D-1$-forms, that belong to a representation $\mathbf{R_{D-1}}$ inside $\mathbf{R_1} \otimes \mathbf{R_{D-2}}$. However, $\mathbf{R_1} \otimes \mathbf{R_{D-2}}$ contains at least three irreducible representations, and the E_{11} derivation of the projection conditions on $\mathbf{R_{D-1}}$ plays an important role in this paper. In particular, it is responsible for the demonstration that the $D-1$-form generators and the embedding tensor belong to the same representation. This was proven in detail for any dimension in this paper.

In table 2 we list all the irreducible representations arising in the tensor product $\mathbf{R_1} \otimes \mathbf{R_0}$ in any dimension, underlying the ones to which the embedding tensor and the $D-1$ forms actually belong. As can be noticed from the table, in four, five and six dimensions $\mathbf{R_1} \otimes \mathbf{R_0}$ generates three representations. These cases are those in which the 1-form generator belongs to the fundamental representation $\mathbf{D_\Lambda}$. Therefore, in four, five and six dimensions the embedding tensor $\Theta_\alpha^{M_1}$ is contained in the tensor product $\mathbf{D_\Lambda} \otimes \mathbf{R_0}$. In the following we will in general denote the adjoint representation by $\mathbf{Adj}$. It is a property of any simple group with the exception of E_8 that the tensor product $\mathbf{D_\Lambda} \otimes \mathbf{Adj}$ always gives

$$\mathbf{D_\Lambda} \otimes \mathbf{Adj} = \mathbf{D_\Lambda} \oplus \mathbf{D_1} \oplus \mathbf{D_2} \quad , \tag{A.8}$$

where $\mathbf{D_1}$ and $\mathbf{D_2}$ are two other representations, and we take the dimension of $\mathbf{D_1}$ to be lower than the dimension of $\mathbf{D_2}$. As can be seen from the table, in four, five and six dimensions the embedding tensor belongs to $\mathbf{D_1}$. Using the fact that this is also the representation of the $D-1$-form, we now show that one can derive from E_{11} the projectors on these three representations. We will focus in particular on the cases of E_7 and E_6, corresponding to four and five dimensions respectively.

Given the tensor product $\mathbf{D_\Lambda} \otimes \mathbf{Adj}$, the projectors $\mathbb{P}_{\mathbf{D_\Lambda}}$, $\mathbb{P}_{\mathbf{D_1}}$ and $\mathbb{P}_{\mathbf{D_2}}$ on the representations of eq. (A.8) can be constructed in terms of δ_N^M, δ_β^α and $D^\alpha_M{}^N$ as [3]

$$\begin{aligned}
\mathbb{P}_{\mathbf{D_\Lambda}}{}_{\alpha N}^{M\beta} &= \frac{d_\Lambda}{d} D^\beta_N{}^P D_{\alpha P}{}^M \\
\mathbb{P}_{\mathbf{D_1}}{}_{\alpha N}^{M\beta} &= a D^\beta_N{}^P D_{\alpha P}{}^M + b D_{\alpha N}{}^P D^\beta_P{}^M + c\delta_N^M \delta_\alpha^\beta \\
\mathbb{P}_{\mathbf{D_2}}{}_{\alpha N}^{M\beta} &= -\left(a + \frac{d_\Lambda}{d}\right) D^\beta_N{}^P D_{\alpha P}{}^M - b D_{\alpha N}{}^P D^\beta_P{}^M + (1-c)\delta_N^M \delta_\alpha^\beta \quad ,
\end{aligned} \tag{A.9}$$

D	G	$\mathbf{R_1} \otimes \mathbf{R_0}$
9	$SL(2,\mathbb{R})$	$\mathbf{1} \oplus \underline{\mathbf{2}} \oplus \mathbf{2} \oplus \underline{\mathbf{3}} \oplus \mathbf{4}$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$(\underline{\mathbf{3},\mathbf{2}}) \oplus (\mathbf{3},\mathbf{2}) \oplus (\mathbf{3},\mathbf{4}) \oplus (\underline{\overline{\mathbf{6}},\mathbf{2}}) \oplus (\mathbf{15},\mathbf{2})$
7	$SL(5,\mathbb{R})$	$\mathbf{10} \oplus \underline{\mathbf{15}} \oplus \underline{\mathbf{40}} \oplus \mathbf{175}$
6	$SO(5,5)$	$\overline{\mathbf{16}} \oplus \underline{\overline{\mathbf{144}}} \oplus \mathbf{560}$
5	$E_{6(+6)}$	$\overline{\mathbf{27}} \oplus \underline{\overline{\mathbf{351}}} \oplus \overline{\mathbf{1728}}$
4	$E_{7(+7)}$	$\mathbf{56} \oplus \underline{\mathbf{912}} \oplus \mathbf{6480}$
3	$E_{8(+8)}$	$\underline{\mathbf{1}} \oplus \mathbf{248} \oplus \underline{\mathbf{3875}} \oplus \mathbf{27000} \oplus \mathbf{30380}$

Table 2. Table giving the irreducible representations that arise in the product $\mathbf{R_1} \otimes \mathbf{R_0}$ in various dimensions. The representations to which the embedding tensor and the $D-1$-form generators belong are underlined.

where one makes use of eq. (A.5) and the fact that the sum of the projectors is the identity. Note that the three coefficients a, b and c are not specified and will be given later. We require the projectors to satisfy

$$\begin{aligned} \mathbb{P}_{\mathbf{D_\Lambda}}{}^{M\beta}_{\alpha N} \mathbb{P}_{\mathbf{D_\Lambda}}{}^{N\gamma}_{\beta P} &= \mathbb{P}_{\mathbf{D_\Lambda}}{}^{M\gamma}_{\alpha P} \qquad \mathbb{P}_{\mathbf{D_\Lambda}}{}^{M\beta}_{\alpha N} \mathbb{P}_{\mathbf{D_i}}{}^{N\gamma}_{\beta P} = 0 \\ \mathbb{P}_{\mathbf{D_i}}{}^{M\beta}_{\alpha N} \mathbb{P}_{\mathbf{D_j}}{}^{N\gamma}_{\beta P} &= \delta_{ij} \mathbb{P}_{\mathbf{D_i}}{}^{M\gamma}_{\alpha P} \qquad i,j = 1,2 \quad . \end{aligned} \tag{A.10}$$

We now show that for E_7 and E_6 these projectors are determined using E_{11}. In principle the E_{11} derivation of the projectors can be also carried out for D_5, which corresponds to the six-dimensional case, but it is not needed because in section 6 we have used the explicit form of the structure constants, which encodes automatically the projectors.

We first consider *the case of* E_7. In section 4 we have shown that the invariant tensor $S_A^{M\alpha}$ resulting from the commutator of the 1-form and the 2-form satisfies the constraints of eqs. (4.13) and (4.14). These constraints follow from the Jacobi identities of the E_{11} algebra. On the other hand, the E_{11} algebra imposes that the 3-form generators in four dimensions belong to the **912**, which implies that the indices $M\alpha$ must be projected on the **912**. This can be seen from the index structure of $S_A^{M\alpha}$ because the only way of building an invariant from tensoring a **912** index with the product $\mathbf{56} \otimes \mathbf{133}$ is that this product is indeed projected on the **912**. By looking at the general form of the projectors in eq. (A.9), we thus must require that $S_A^{M\alpha}$ satisfies the conditions

$$\begin{aligned} \mathbb{P}_{\mathbf{56}}{}^{M\beta}_{\alpha N} S_A^{N\gamma} g_{\beta\gamma} &= 0 \\ \mathbb{P}_{\mathbf{912}}{}^{M\beta}_{\alpha N} S_A^{N\gamma} g_{\beta\gamma} &= S_A^{M\gamma} g_{\alpha\gamma} \\ \mathbb{P}_{\mathbf{6480}}{}^{M\beta}_{\alpha N} S_A^{N\gamma} g_{\beta\gamma} &= 0 \quad , \end{aligned} \tag{A.11}$$

and comparing these three conditions with eqs. (4.13) and (4.14) we determine a constraint on the parameters a, b and c in eq. (A.9). Indeed, the first condition is automatically satisfied because it reproduces eq. (4.13), while the second and the third reproduce eq. (4.14) provided that

$$\frac{b}{1-c} = -2 \quad . \tag{A.12}$$

We now derive the constraints resulting from eq. (A.10) in case of E_7. The first condition is automatically satisfied, while the second condition gives

$$\frac{19}{8}a + \frac{7}{8}b + c = 0 \quad , \tag{A.13}$$

where we have made use of eq. (A.6) in which we have substituted the dual Coxeter number and the Dynkin index for E_7, which are listed in table 3. In order to derive the other constraints, one makes use of the identities

$$D_{\alpha N}{}^{M} D_{\beta MP} D^{\beta NQ} = \frac{7}{8} D_{\alpha P}{}^{Q} \quad , \tag{A.14}$$

$$D_{\beta M}{}^{N} D_{P}^{\beta\, Q} = \frac{1}{12}\delta_M^Q \delta_P^N + \frac{1}{24}\delta_M^N \delta_P^Q - \frac{1}{24}\Omega^{NQ}\Omega_{MP} + D_\beta^{NQ} D_{MP}^\beta \quad , \tag{A.15}$$

and

$$(D^\gamma D_\alpha)_Q{}^R D_P^{\beta\, Q} D_{\beta R}{}^M = \frac{1}{24}\delta_P^M \delta_\alpha^\gamma - \frac{3}{8}(D^\gamma D_\alpha)_P{}^M - \frac{5}{12}(D_\alpha D^\gamma)_P{}^M \quad , \tag{A.16}$$

which can all be proven using the results listed in this appendix and in section 4. Using these results, the last condition in eq. (A.10), with $i = j = 1$, gives

$$\begin{aligned} \frac{19}{8}a^2 + \frac{7}{4}ab + 2ac - \frac{3}{8}b^2 &= a \\ 2bc - \frac{5}{12}b^2 &= b \\ c^2 + \frac{1}{24}b^2 &= c \quad . \end{aligned} \tag{A.17}$$

Substituting eq. (A.12) into the last of these equations gives

$$c = \frac{1}{7} \qquad b = -\frac{12}{7} \quad , \tag{A.18}$$

and from eq. (A.13) one gets $a = \frac{4}{7}$. One can show that all the other projector conditions are satisfied. Substituting this in eq. (A.9) finally gives the E_7 projectors

$$\begin{aligned} \mathbb{P}_{\mathbf{56}}{}^{M\beta}_{\alpha N} &= \frac{8}{19} D_N^{\beta\, P} D_{\alpha P}{}^M \\ \mathbb{P}_{\mathbf{912}}{}^{M\beta}_{\alpha N} &= \frac{4}{7} D_N^{\beta\, P} D_{\alpha P}{}^M - \frac{12}{7} D_{\alpha N}{}^P D_P^{\beta\, M} + \frac{1}{7}\delta_N^M \delta_\alpha^\beta \\ \mathbb{P}_{\mathbf{6480}}{}^{M\beta}_{\alpha N} &= -\frac{132}{133} D_N^{\beta\, P} D_{\alpha P}{}^M + \frac{12}{7} D_{\alpha N}{}^P D_P^{\beta\, M} + \frac{6}{7}\delta_N^M \delta_\alpha^\beta \quad . \end{aligned} \tag{A.19}$$

We now derive from E_{11} the projectors of eq. (A.9) for **the case of** E_6. We first derive the identities that will be needed. Is section 5 we have introduced the completely symmetric invariant tensors d^{MNP} and d_{MNP}, satisfying

$$d^{MNP} d_{MNQ} = \delta_Q^P \quad . \tag{A.20}$$

From eq. (A.20) and the condition that d^{MNP} and d_{MNP} are invariant tensors,

$$D^{\alpha}_{M}{}^{(N}d^{PQ)M} = D^{\alpha}_{(M}{}^{N}d_{PQ)N} = 0 \quad , \tag{A.21}$$

one gets

$$D^{\alpha}_{M}{}^{N}d^{MPQ}d_{NSQ} = -\frac{1}{2}D^{\alpha Q}_{S} \quad . \tag{A.22}$$

One can write the product of two generators in the **27** contracted by the metric $g_{\alpha\beta}$ as

$$g_{\alpha\beta}D^{\alpha}_{M}{}^{N}D^{\beta Q}_{P} = \alpha\delta^{N}_{P}\delta^{Q}_{M} + \beta\delta^{N}_{M}\delta^{Q}_{P} + \gamma d^{NQR}d_{MPR} \quad , \tag{A.23}$$

as can be deduced from the fact that the product $\mathbf{27}\otimes\mathbf{27}\otimes\overline{\mathbf{27}}\otimes\overline{\mathbf{27}}$ leads to three different E_6 invariants, and the three invariant quantities on the right hand side of eq. (A.23) are the most general objects one can write down in terms of δ^{N}_{M}, d^{MNP} and d_{MNP}. Eq. (A.5), applied to the E_6 case in which $d = 78$ and $d_\Lambda = 27$, is

$$g_{\alpha\beta}D^{\alpha}_{M}{}^{N}D^{\beta}_{N}{}^{P} = \frac{26}{9}\delta^{P}_{M} \quad . \tag{A.24}$$

Contracting N and P in eq. (A.23) thus leads to

$$27\alpha + \beta + \gamma = \frac{26}{9} \quad , \tag{A.25}$$

while contracting M and N gives

$$\alpha + 27\beta + \gamma = 0 \quad . \tag{A.26}$$

A third relation comes from the identity

$$g_{\alpha\beta}D^{\alpha}_{M}{}^{N}D^{\beta Q}_{P}d_{NQR} = -\frac{13}{9}d_{MPR} \quad , \tag{A.27}$$

which can be derived using eq. (A.21) iteratively, and leads to

$$\alpha + \beta + \gamma = -\frac{13}{9} \quad , \tag{A.28}$$

or alternatively from contracting eq. (A.23) with $D^{\gamma}_{N}{}^{M}$, which leads to

$$\alpha - \frac{1}{2}\gamma = 1 \quad . \tag{A.29}$$

The final result is

$$\alpha = \frac{1}{6} \qquad \beta = \frac{1}{18} \qquad \gamma = -\frac{5}{3} \quad , \tag{A.30}$$

so that

$$g_{\alpha\beta}D^{\alpha}_{M}{}^{N}D^{\beta Q}_{P} = \frac{1}{6}\delta^{N}_{P}\delta^{Q}_{M} + \frac{1}{18}\delta^{N}_{M}\delta^{Q}_{P} - \frac{5}{3}d^{NQR}d_{MPR} \quad . \tag{A.31}$$

Another useful relation is

$$(D^{\gamma}D_{\alpha})_{Q}{}^{R}d^{QMT}d_{PRT} = -\frac{1}{5}(D^{\gamma}D_{\alpha})_{P}{}^{M} + \frac{3}{10}(D_{\alpha}D^{\gamma})_{P}{}^{M} + \frac{1}{30}\delta^{M}_{P}\delta^{\gamma}_{\alpha} \quad , \tag{A.32}$$

which can be derived using the relations given in this appendix.

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In section 5 we have shown that the invariant tensor $S^{\alpha M,NP}$ resulting from the commutator of the 1-form and the 3-form satisfies the constraints of eqs. (5.9) and (5.13). These constraints follow from requiring the closure of the Jacobi identities of the E_{11} algebra. On the other hand, the form of this invariant tensor is dictated by the fact that E_{11} imposes that the 4-form generator is in the $\overline{\mathbf{351}}$, and therefore the indices αM of $\overline{\mathbf{351}}$ must be projected on the $\overline{\mathbf{351}}$. This can be seen from the index structure of $S^{\alpha M,NP}$ because the NP antisymmetric indices correspond to the $\mathbf{351}$ and the only way of building an invariant from tensoring a $\mathbf{351}$ representation with the product $\overline{\mathbf{27}} \otimes \mathbf{78}$ is that this product is indeed projected on the $\overline{\mathbf{351}}$. By looking at the general expressions of eq. (A.9) for the projectors, we therefore must impose the conditions

$$\begin{aligned} \mathbb{P}_{\overline{\mathbf{27}}\alpha N}{}^{M\beta} S_A^{N\gamma} g_{\beta\gamma} &= 0 \\ \mathbb{P}_{\overline{\mathbf{351}}\alpha N}{}^{M\beta} S_A^{N\gamma} g_{\beta\gamma} &= S_A^{M\gamma} g_{\alpha\gamma} \\ \mathbb{P}_{\overline{\mathbf{1728}}\alpha N}{}^{M\beta} S_A^{N\gamma} g_{\beta\gamma} &= 0 \quad , \end{aligned} \tag{A.33}$$

and comparing these three conditions with eqs. (5.9) and (5.13) we determine a constraint on the parameters a, b and c in eq. (A.9). In particular, the first condition is automatically satisfied because it reproduces eq. (5.9), while eq. (5.13) implies that the second and the third equations give the same constraint, that is

$$\frac{b}{1-c} = -\frac{3}{2} \quad . \tag{A.34}$$

We now derive the constraints resulting from eq. (A.10) in case of E_6. The first condition is automatically satisfied, while the second condition gives

$$\frac{26}{9}a + \frac{8}{9}b + c = 0 \quad , \tag{A.35}$$

where we have made use of eq. (A.6) in which we have substituted the dual Coxeter number and the Dynkin index for E_6, which are listed in table 3. The last condition in eq. (A.10), with $i = j = 1$, gives

$$\begin{aligned} \frac{26}{9}a^2 + \frac{16}{9}ab + 2ac + \frac{7}{18}b^2 &= a \\ 2bc - \frac{1}{2}b^2 &= b \\ c^2 + \frac{1}{9}b^2 &= c \quad . \end{aligned} \tag{A.36}$$

Substituting eq. (A.34) into the last of these equations gives

$$c = \frac{1}{5} \qquad b = -\frac{6}{5} \quad , \tag{A.37}$$

and from eq. (A.35) one gets $a = \frac{3}{10}$. One can show that all the other projector conditions are satisfied. Substituting this in eq. (A.9) finally gives the E_6 projectors

$$\begin{aligned} \mathbb{P}_{\overline{\mathbf{27}}\alpha N}{}^{M\beta} &= \frac{9}{26} D_N^{\beta\,P} D_{\alpha P}{}^M \\ \mathbb{P}_{\overline{\mathbf{351}}\alpha N}{}^{M\beta} &= \frac{3}{10} D_N^{\beta\,P} D_{\alpha P}{}^M - \frac{6}{5} D_{\alpha N}{}^P D_P^{\beta\,M} + \frac{1}{5}\delta_N^M \delta_\alpha^\beta \\ \mathbb{P}_{\overline{\mathbf{1728}}\alpha N}{}^{M\beta} &= -\frac{45}{65} D_N^{\beta\,P} D_{\alpha P}{}^M + \frac{6}{5} D_{\alpha N}{}^P D_P^{\beta\,M} + \frac{4}{5}\delta_N^M \delta_\alpha^\beta \quad . \end{aligned} \tag{A.38}$$

G	$g^\vee$	$\tilde{I}_\Lambda$	d	d_Λ	d_1	a	b	c
A_r	$r+1$	$\frac{1}{2}$	r^2+2r	$r+1$	$\frac{1}{2}(r-1)(r+1)(r+2)$	$-\frac{1}{2r}$	$-\frac{1}{2}$	$\frac{1}{2}$
G_2	4	1	14	7	27	$-\frac{3}{14}$	$-\frac{6}{7}$	$\frac{3}{7}$
F_4	9	3	52	26	273	$\frac{1}{4}$	$-\frac{3}{2}$	$\frac{1}{4}$
E_6	12	3	78	27	351	$\frac{3}{10}$	$-\frac{6}{5}$	$\frac{1}{5}$
E_7	18	6	133	56	912	$\frac{4}{7}$	$-\frac{12}{7}$	$\frac{1}{7}$

Table 3. Table giving the dual Coxeter number, the Dynkin index, the dimension of the adjoint, the fundamental and the $\mathbf{D_1}$ representations, as well as the parameters a, b and c occurring in eq. (A.9), for some simple Lie groups (see also [3]).

The projectors of eqs. (A.19) and (A.38) that we have obtained from E_{11} exactly coincide with the projectors that one obtains from group theory. In table 3 we list the dual Coxeter number, the Dynkin index of the fundamental representation, the dimension of the group d, the dimension of the fundamental representation d_Λ and the dimension d_1 of the representation $\mathbf{D_1}$, as well as the values of the coefficients a, b and c in eq. (A.9), for some simple Lie groups. One can see in particular that the values of a, b and c in the table for E_6 and E_7 are exactly those that we have derived from E_{11}.

B Field strengths and gauge transformations

In this appendix we explicitly evaluate the deformed part of the field strengths up to rank six from eqs. (2.46) and (2.48). For the scalar derivative we find

$$F_a = g_\varphi^{-1}(\partial_a + gA_{a,N_1}\Theta_\alpha^{N_1}R^\alpha)g_\varphi \quad . \tag{B.1}$$

We now write down the field strengths for the gauge fields, which by assumption have all their Lorentz indices anti-symmetrised and as discussed in section two we do not explicitly display their scalar factors converting from a linear representation to a non-linear representation of G. We first display the massless part of the field strengths up to rank 6 included, determined using eq. (2.46). The result is

$$F^{(0)}_{ca_1,N_1} = 2\partial_c A_{a_1,N_1} \tag{B.2}$$
$$F^{(0)}_{ca_1a_2,N_2} = 3\left[\partial_c A_{a_1a_2,N_2} - \frac{1}{2}L_{a_1,N_2}{}^{N_1}\partial_c A_{a_2,N_1}\right] \tag{B.3}$$
$$F^{(0)}_{ca_1a_2a_3,N_3} = 4\left[\partial_c A_{a_1a_2a_3,N_3} - L_{a_1,N_3}{}^{N}_{2}\partial_c A_{a_2a_3,N_2} + \frac{1}{3!}(L_{a_1}L_{a_2})_{N_3}{}^{N_1}\partial_c A_{a_3,N_1}\right] \tag{B.4}$$

$$F^{(0)}_{ca_1\ldots a_4,N_4} = 5\Big[\partial_c A_{a_1\ldots a_4,N_4} + L_{a_1,N_4}{}^{N_3}\partial_c A_{a_2a_3a_4,N_3} + \frac{1}{2}(L_{a_1}L_{a_2})_{N_4}{}^{N_2}\partial_c A_{a_3a_4,N_2}$$
$$-\frac{1}{2}L_{a_1a_2,N_4}{}^{N_2}\partial_c A_{a_3a_4,N_2} - \frac{1}{4!}(L_{a_1}L_{a_2}L_{a_3})_{N_4}{}^{N_1}\partial_c A_{a_4,N_1}\Big] \tag{B.5}$$

$$F^{(0)}_{ca_1\ldots a_5,N_5} = 6\Big[\partial_c A_{a_1\ldots a_5,N_5} - L_{a_1,N_5}{}^{N_4}\partial_c A_{a_2\ldots a_5,N_4} - L_{a_1a_2,N_5}{}^{N_3}\partial_c A_{a_3a_4a_5,N_3}$$
$$+\frac{1}{2}(L_{a_1}L_{a_2})_{N_5}{}^{N_3}\partial_c A_{a_3a_4a_5,N_3} - \frac{1}{3!}(L_{a_1}L_{a_2}L_{a_3})_{N_5}{}^{N_2}\partial_c A_{a_4a_5,N_2}$$
$$+\frac{1}{2}(L_{a_1}L_{a_2a_3})_{N_5}{}^{N_2}\partial_c A_{a_4a_5,N_2} + \frac{1}{5!}(L_{a_1}L_{a_2}L_{a_3}L_{a_4})_{N_5}{}^{N_1}\partial_c A_{a_5,N_1}\Big]. \tag{B.6}$$

The order g part of the same field strengths follows from eq. (2.48). The result is

$$F^{(1)}_{ca_1,N_1} = 2g\Big[W^{N_2}{}_{N_1}A_{ca_1,N_2} + \frac{1}{2}X^{M_1P_1}{}_{N_1}A_{a_1,P_1}A_{c,M_1}\Big] \tag{B.7}$$

$$F^{(1)}_{ca_1a_2,N_2} = 3g\Big[W^{N_3}{}_{N_2}A_{ca_1a_2,N_3} - L_{a_1,N_2}{}^{N_1}W^{M_2}{}_{N_1}A_{ca_2,M_2}$$
$$-\frac{1}{3!}L_{a_1,N_2}{}^{P_1}X^{N_1M_1}{}_{P_1}A_{a_2,M_1}A_{c,N_1}\Big] \tag{B.8}$$

$$F^{(1)}_{ca_1a_2a_3,N_3} = 4g\Big[W^{N_4}{}_{N_3}A_{ca_1a_2a_3,N_4} - L_{a_1,N_3}{}^{N_2}W^{M_3}{}_{N_2}A_{ca_2a_3,M_3}$$
$$-\frac{1}{2}L_{a_1a_2,N_3}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_3,N_2} + \frac{1}{2}(L_{a_1}L_{a_2})_{N_3}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_3,N_2}$$
$$+\frac{1}{4!}(L_{a_1}L_{a_2})_{N_3}{}^{P_1}X^{N_1M_1}{}_{P_1}A_{a_3,M_1}A_{c,N_1}\Big] \tag{B.9}$$

$$F^{(1)}_{ca_1\ldots a_4,N_4} = 5g\Big[W^{N_5}{}_{N_4}A_{ca_1\ldots a_4,N_5} - L_{a_1,N_4}{}^{N_3}W^{M_4}{}_{N_3}A_{ca_2a_3a_4,M_4}$$
$$-L_{a_1a_2,N_4}{}^{N_2}W^{N_3}{}_{N_2}A_{ca_3a_4,N_3} + \frac{1}{2}(L_{a_1}L_{a_2})_{N_4}{}^{N_2}W^{N_3}{}_{N_2}A_{ca_3a_4,N_3}$$
$$+\frac{1}{2}(L_{a_1}L_{a_2a_3})_{N_4}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_4,N_2} - \frac{1}{3!}(L_{a_1}L_{a_2}L_{a_3})_{N_4}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_4,N_2}$$
$$-\frac{1}{5!}(L_{a_1}L_{a_2}L_{a_3})_{N_4}{}^{P_1}X^{N_1M_1}{}_{P_1}A_{a_4,M_1}A_{c,N_1}\Big] \tag{B.10}$$

$$F^{(1)}_{ca_1\ldots a_5,N_5} = 6g\Big[W^{N_6}{}_{N_5}A_{ca_1\ldots a_5,N_6} - L_{a_1,N_5}{}^{N_4}W^{M_5}{}_{N_4}A_{ca_2\ldots a_5,M_5}$$
$$-L_{a_1a_2,N_5}{}^{N_3}W^{N_4}{}_{N_3}A_{ca_3\ldots a_5,N_4} + \frac{1}{2}(L_{a_1}L_{a_2})_{N_5}{}^{N_3}W^{N_4}{}_{N_3}A_{ca_3\ldots a_5,N_4}$$
$$-\frac{1}{2}L_{a_1a_2a_3,N_5}{}^{N_2}W^{N_3}{}_{N_2}A_{ca_4a_5,N_3} + (L_{a_1}L_{a_2a_3})_{N_5}{}^{N_2}W^{N_3}{}_{N_2}A_{ca_4a_5,N_3}$$
$$-\frac{1}{3!}(L_{a_1}L_{a_2}L_{a_3})_{N_5}{}^{N_2}W^{N_3}{}_{N_2}A_{ca_4a_5,N_3} + \frac{1}{3!}(L_{a_1a_2}L_{a_3a_4})_{N_5}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_5,N_2}$$
$$-\frac{1}{4}(L_{a_1}L_{a_2}L_{a_3a_4})_{N_5}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_5,N_2} + \frac{1}{4!}(L_{a_1}L_{a_2}L_{a_3}L_{a_4})_{N_5}{}^{N_1}W^{N_2}{}_{N_1}A_{ca_5,N_2}$$
$$+\frac{1}{6!}(L_{a_1}L_{a_2}L_{a_3}L_{a_4})_{N_5}{}^{P_1}X^{N_1M_1}{}_{P_1}A_{a_5,M_1}A_{c,N_1}\Big] \quad . \tag{B.11}$$

The reader can easily evaluate the remaining field strengths.

The rigid transformations of the group element also determine the gauge transformations of the various fields. We list here the gauge transformations for all the forms up to the 6-form. The 1-form transforms as

$$\delta A_{a_1,N_1} = a_{a_1,N_1} - g\Lambda_{M_1} X_1^{M_1}{}_{N_1}{}^{P_1} A_{a_1,P_1} \quad , \tag{B.12}$$

the 2-form as

$$\delta A_{a_1a_2,N_2} = a_{a_1a_2,N_2} - \frac{1}{2} A_{a_1,N_1} a_{a_2,M_1} f^{N_1M_1}{}_{N_2} - g\Lambda_{M_1} X_2^{M_1}{}_{N_2}{}^{P_2} A_{a_1a_2,P_2} \quad , \tag{B.13}$$

the 3-form as

$$\begin{aligned} \delta A_{a_1a_2a_3,N_3} = {} & a_{a_1a_2a_3,N_3} - A_{a_1a_2,N_2} a_{a_3,N_1} f^{N_2N_1}{}_{N_3} \\ & - \frac{1}{3!} A_{a_1,N_1} A_{a_2,M_1} a_{a_3,P_1} f^{N_1N_2}{}_{N_3} f^{M_1P_1}{}_{N_2} - g\Lambda_{M_1} X_3^{M_1}{}_{N_3}{}^{P_3} A_{a_1a_2a_3,P_3} \quad , \end{aligned} \tag{B.14}$$

the 4-form transforms as

$$\begin{aligned} \delta A_{a_1\dots a_4,N_4} = {} & a_{a_1\dots a_4,N_4} - \frac{1}{2} A_{a_1a_2,N_2} a_{a_3a_4,M_2} f^{N_2M_2}{}_{N_4} - A_{a_1a_2a_3,N_3} a_{a_4,N_1} f^{N_3N_1}{}_{N_4} \\ & - \frac{1}{4!} A_{a_1,N_1} A_{a_2,M_1} A_{a_3,P_1} a_{a_4,Q_1} f^{N_1N_3}{}_{N_4} f^{M_1N_2}{}_{N_3} f^{P_1Q_1}{}_{N_2} \\ & - \frac{1}{4} A_{a_1a_2,N_2} A_{a_3,N_1} a_{a_4,M_1} f^{N_2M_2}{}_{N_4} f^{N_1M_1}{}_{M_2} - g\Lambda_{M_1} X_4^{M_1}{}_{N_4}{}^{P_4} A_{a_1\dots a_4,P_4} \, , \end{aligned} \tag{B.15}$$

and the 5-form transforms as

$$\begin{aligned} \delta A_{a_1\dots a_5,N_5} = {} & a_{a_1\dots a_5,N_5} - A_{a_1\dots a_3,N_3} a_{a_4a_5,N_2} f^{N_3N_2}{}_{N_5} - A_{a_1\dots a_4,N_4} a_{a_5,N_1} f^{N_4N_1}{}_{N_5} \\ & - \frac{1}{2} A_{a_1a_2,N_2} A_{a_3a_4,M_2} a_{a_5,N_1} f^{N_2N_3}{}_{N_5} f^{M_2N_1}{}_{N_3} \\ & - \frac{1}{5!} A_{a_1,N_1} A_{a_2,M_1} A_{a_3,P_1} A_{a_4,Q_1} a_{a_5,R_1} f^{N_1N_4}{}_{N_5} f^{M_1N_3}{}_{N_4} f^{P_1N_2}{}_{N_3} f^{Q_1R_1}{}_{N_2} \\ & - \frac{1}{3!} A_{a_1a_2,N_2} A_{a_3,N_1} A_{a_4,M_1} a_{a_5,P_1} f^{N_2N_3}{}_{N_5} f^{N_1M_2}{}_{N_3} f^{M_1P_1}{}_{M_2} \\ & - g\Lambda_{M_1} X_5^{M_1}{}_{N_5}{}^{P_5} A_{a_1\dots a_5,P_5} \; . \end{aligned} \tag{B.16}$$

The parameters a are given in eq. (2.52) in terms of the gauge parameters Λ. The reader can easily evaluate the gauge transformations for the higher rank fields.

C Extended spacetime in four dimensions

In this appendix we will consider the four dimensional maximal gauged supergravities using a non-linear realisation of $E_{11} \otimes_s l_1$. This closely follows the similar derivation of the maximal gauged supergravities in five dimensions given in reference [16] to which we refer for the details of how this method works.

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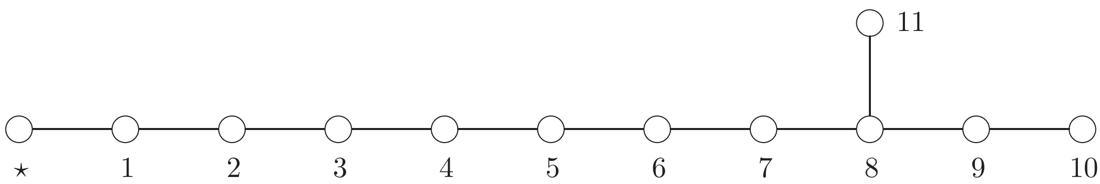

Figure 8. The E_{12} Dynkin diagram.

C.1 The l_1 multiplet in four dimensions

The l_1 multiplet can be thought of as the E_{11} representation that contains the momentum generator P_c as its lowest component. The l_1 multiplet is the representation of E_{11} with highest weight λ_1, where λ_1 is the fundamental weight associated with node 1 of the Dynkin diagram of E_{11}. By definition it satisfies the relation $(\lambda_1, \alpha_i) = \delta_{1i}$, where α_i is the simple root associated with node i on the Dynkin diagram of E_{11}. For our derivation we will need the l_1 multiplet of E_{11} suitable to four dimensions at low levels.

The most straightforward way to find the components of the l_1 multiplet as it occurs in four dimensions at low levels is just to take the l_1 multiplet in eleven dimensions [15], carry out the dimensional reduction to four dimensions by hand, and then collect the result into representations of the internal symmetry group E_7. A more sophisticated method is to realise that the l_1 representation of E_{11} can be obtained by considering the adjoint representation of E_{12}. The Dynkin diagram of E_{12} is just that of E_{11}, but with one node, the starred node, added with one line attached to node one as in figure 8. To find the l_1 representation suitable to eleven dimensions we decompose the adjoint representation of E_{12} into representations of the E_{11} obtained by deleting the starred node in the E_{12} Dynkin diagram and keeping only the level one generators; by level we mean the level associated with node one [15]. Clearly, as the commutation relations respect the level we must find a representation of E_{11} and it is in fact the l_1 representation. To find the l_1 representation in four dimensions one then carries out the decomposition $GL(4,\mathbb{R}) \otimes E_7$ corresponding to deleting in addition node four.

Following either method the low level elements of the l_1 multiplet in four dimensions are found to be given by [37, 38]

$$\begin{aligned} &P_a\ (\mathbf{1}) \quad Z^M\ (\mathbf{56}) \quad Z^{a,\alpha}\ (\mathbf{133} \oplus \mathbf{1}) \quad Z^{ab,N\beta}\ (\mathbf{912} \oplus \mathbf{56} \oplus \mathbf{1}) \\ &\qquad Z^{abc,[\delta\epsilon]}\ (\mathbf{8645} \oplus \mathbf{1539} \oplus \mathbf{133} \oplus \mathbf{1}) \quad \cdots \end{aligned} \tag{C.1}$$

The indices $a, b, \ldots = 0, 1, 2, 3$ transform under $GL(4,\mathbb{R})$ in the obvious way while the numbers in brackets indicate the dimensions of the E_7 representations to which the charges belong.

The commutators of E_{11} appropriate to four dimensions are given in section four. The commutators of the E_7 generators with the l_1 generators are determined by the E_7

representation that the charges belong to and are given by

$$\begin{aligned}[R^\alpha, Z^M] &= (D^\alpha)_N{}^M Z^N \\ [R^\alpha, Z^{a,\beta}] &= f^{\alpha\beta}{}_\gamma Z^{a,\gamma} \\ [R^\alpha, Z^{ab,N\beta}] &= (D^\alpha)_M{}^N Z^{ab,M\beta} + f^{\alpha\beta}{}_\gamma Z^{ab,N\gamma} \\ [R^\alpha, Z^{abc,[\delta\epsilon]}] &= f^{\alpha\delta}{}_\beta Z^{abc,[\beta\epsilon]} + f^{\alpha\epsilon}{}_\beta Z^{abc,[\delta\beta]} \quad .\end{aligned} \tag{C.2}$$

The remaining commutators between E_{11} generators and those of the l_1 multiplet can be deduced from their E_{12} origin, or just writing down relations compatible with the level assignments, $GL(4,\mathbb{R})$ character, and using the Jacobi identities. We may define the way the l_1 generators occur in the $E_{11} \otimes_s l_1$ algebra by the relations

$$\begin{aligned} &[R^\alpha, P_a] = 0, \qquad [R^{a,N}, P_b] = \delta^a_b Z^N, \qquad [R^{ab,\alpha}, P_c] = 2\delta^{[a}_c Z^{b],\alpha} \\ &[R^{abc,N\beta}, P_d] = 3\delta^{[a}_d Z^{bc],N\beta}, \quad [R^{abcd,[\alpha\beta]}, P_e] = 4\delta^{[a}_e Z^{bcd],[\alpha\beta]} \quad . \end{aligned} \tag{C.3}$$

The normalisation of the l_1 generators is then fixed by the choice of coefficients on the right hand side. The fact that the representation of the charges and the E_{11} generators coincide on each side of these equations is a consequence of the relationship that exists between the l_1 representation and the adjoint representation of E_{11} [37]. Physically this is the usual relationship between fields and the charges to which they couple in the Wess-Zumino term of a brane action.

The remainder of the commutators may be fixed through the Jacobi identities. For example, let us consider the Jacobi identity involving $[R^{a,M}, [R^{b,N}, P_c]]$. We find, using the E_{11} commutators of eq. (4.9), that

$$[R^{a,M}, Z^N] = -D^{MN}_\alpha Z^{a,\alpha} \quad . \tag{C.4}$$

Now, for convenience we define $Z^{ab,N\beta} = S^{N\beta}_A Z^{ab,A}$, and the Jacobi relation $[R^{a,M}, [R^{bc,\alpha}, P_d]]$ implies that

$$[R^{ab,\alpha}, Z^M] = -Z^{ab,M\alpha} \quad \text{and} \quad [R^{a,M}, Z^{b,\alpha}] = -Z^{ab,M\alpha} \quad . \tag{C.5}$$

The Jacobi relation $[R^{ab,\alpha}, [R^{cd,\beta}, P_e]]$ leads to

$$[R^{ab,\alpha}, Z^{c,\beta}] = Z^{abc,[\alpha\beta]} \quad . \tag{C.6}$$

The final Jacobi identity $[R^{abc,A}, [R^{d,M}, P_e]]$ implies both

$$[R^{a,M}, Z^{bc,A}] = -c^{MA}_{\alpha\beta} Z^{abc,[\alpha\beta]} \quad \text{and} \quad [R^{abc,A}, Z^M] = -c^{MA}_{\alpha\beta} Z^{abc,[\alpha\beta]} \quad . \tag{C.7}$$

Thus the commutators between the E_{11} generators and those of l_1 at low levels are given in equations (C.3) to (C.7).

C.2 Construction of the four dimensional gauged maximal supergravities

The field strengths and gauge transformations of the massless theory follow in a straightforward way from the E_{11} algebra, they are essentially the Cartan forms subject to the appropriate anti-symmetrisation. They are given by the general formula in eq. (2.46) and more explicitly in section four by setting $g = 0$. We will now derive the field strengths for the gauged theory following closely the argument given in [16] for the five-dimensional case.

We begin by choosing the group element of the non-linear realisation of $E_{11} \otimes_s l_1$ to be

$$g = g_{l_1} g_A = e^{x^a P_a} e^{y \cdot Y} e^{A(x) \cdot R} \quad , \tag{C.8}$$

where

$$e^{A(x) \cdot R} = e^{A_{a_1 \dots a_4, \alpha\beta} R^{a_1 \dots a_4, \alpha\beta}} \dots e^{A_{a,M} R^{a,M}} e^{\phi_\alpha R^\alpha} \tag{C.9}$$

and

$$e^{y \cdot Y} = e^{y_M (Z^M + g T^M)} e^{y_{a,\alpha}(Z^{a,\alpha} + g T^{a,\alpha})} e^{y_{ab,M\alpha}(Z^{ab,M\alpha} + g T^{ab,M\alpha})} \dots \quad . \tag{C.10}$$

The symbols in this latter equation are defined by

$$\begin{aligned} T^M &= \Theta_\alpha^M R^\alpha, & T^{a,\alpha} &= W_M^\alpha R^{a,M} \\ T^{ab,M\alpha} &= \frac{1}{3}(\Theta_\beta^M f^{\beta\alpha}{}_\gamma - 2 W_N^\alpha D_\gamma^{MN}) R^{ab,\gamma}, & T^{abc,[\epsilon\delta]} &= W_M^\delta R^{abc,M\epsilon}, \dots \quad . \end{aligned} \tag{C.11}$$

The coefficients in these expressions are taken from the results of section 4. The tensor W_M^α defined here differs from the tensor denoted in the same way in section 4 by a factor of 2, and indeed it is defined as

$$W_M^\alpha = -\frac{1}{2} \Theta_M^\alpha \tag{C.12}$$

which differs from eq. (4.25). Again, this coefficient is taken to reproduce the results of section 4.

At first sight it may appear that the above group element contains all the generators of the l_1 multiplet, but this is not the case. In fact the coordinates y obey projections conditions that mean that part of the l_1 multiplet is absent and plays no role in the calculation. As discussed in reference [16], the part of l_1 which is present is the image of a map from E_{11} into the l_1 representation. It is argued [16] that demanding certain natural properties of this map leads to the constraints on Θ_α^M, etc found as a consequence of Jacobi identities of the deformed algebra in this paper. These constraints on Θ etc are reflected in the projections arising from the y coordinates. One may hope that such an argument may fix the coefficients in eqs. (C.11) and (C.12).

For reasons to do with the preservation of the form of the group element under the action of the group and the required compensating transformations the field strengths of the gauged supergravity are not given in a simple way by the Cartan forms. However, as explained in reference [16] we can calculate the variation of the fields in the usual way by taking a group element g_0 and considering its effect on the coset representative $g_0 g \longrightarrow g'$ and find the field strengths by demanding that they be invariant under these transformations. In particular let us consider $g \to g_0 g$ taking $g_0 = e^{b \cdot Z}$ where the parameter b obeys the same constraints as the y coordinates. As discussed in reference [16] $e^{b \cdot Z} e^{y \cdot Y} =$

$e^{y'\cdot Z}e^{-gb\cdot T}$. However, as the final field strengths do not depend on the y coordinates we need only calculate the the effect of $e^{-gb\cdot T}$ on the $e^{A\cdot R}$ term and do not require to know how the y coordinates change.

Let us first consider the transformation $g \to g_0 g$ with $g_0 = e^{b_M Z^M}$ this leads to the factor $e^{-gb_M \Theta_\alpha^M R^\alpha}$ acting on the E_{11} coset representative which is just the same as an E_{11} transformation of the ungauged theory, but with parameter $a_\alpha = b_M \Theta_\alpha^M$. At the next level we consider the transformation $g_0 = e^{b_{a,\alpha} Z^{a,\alpha}}$, which results in the factor $e^{-gb_{a,\alpha}T^{a,\alpha}} = e^{-gb_{a,\alpha}W_M^\alpha R^{a,M}}$ which is just an E_{11} transformation with the parameter $a_{a,M} = -gb_{a,\alpha}W_M^\alpha$. Similarly the effect of $g_0 = e^{b_{ab,M\alpha}Z^{ab,M\alpha}}$ is equivalent to an E_{11} transformation with parameter $a_{ab,\alpha} = -gb_{ab,M\beta}\frac{1}{3}(\Theta_\gamma^M f^{\gamma\beta}{}_\alpha - 2W_N^\beta D_\alpha^{MN})$ and $g_0 = e^{b_{abc,[\epsilon\delta]}V^{abc,[\epsilon\delta]}}$ is equivalent to an E_{11} transformation with parameter $a_{abc,A} = -gb_{abc,[\epsilon\delta]}W_M^\delta S_A^{M\epsilon}$. The result of all these transformations on the fields is given by

$$\begin{aligned}
\delta A_{a,M} &= -gb_P \Theta_\alpha^P (D^\alpha)_M{}^N A_{a,N} \\
\delta A_{a_1 a_2,\alpha} &= -gb_P \Theta_\beta^P f^{\beta\alpha}{}_\gamma A_{a_1 a_2,\alpha} \\
\delta A_{a_1\ldots a_3,M\alpha} &= -gb_P \Theta_\beta^P (D_M^\beta{}^N A_{a_1\ldots a_3,N\alpha} + f^{\beta\gamma}{}_\alpha A_{a_1\ldots a_3,M\gamma}) \quad .
\end{aligned} \tag{C.13}$$

We now consider an E_{11} transformations for the gauged theory, that is we take $g_0 = e^{a\cdot R}$ to act on the group element of eq. (C.8);

$$g_0 g = e^{a\cdot R}e^{x^a P_a}e^{y\cdot Y}e^{A\cdot R} = e^{x^a P_a + [a\cdot R, x^a P_a]}e^{y\cdot Y + [a\cdot R, y\cdot Y]}e^{a\cdot R}e^{A\cdot R} \quad . \tag{C.14}$$

As discussed in [16] the transformations resulting from the $[R, Y]$ in the second term do not affect the dynamics as the final dynamics does not depend on the y coordinates and so we may ignore this term. The final $a \cdot R$ term has the same effect on the E_{11} fields as the equivalent transformation in the ungauged theory. In the first factor we find $e^{x^a P_a + [a\cdot R, x^a P_a]}$ which leads to higher generators in the l_1 multiplet. For example, if we take $g_0 = e^{a_{a,M}R^{a,M}}$ we find it leads to the term $e^{x^b P_b}e^{x^a a_{a,M}Z^M}$. This latter factor then acts like a l_1 transformation on the rest of the coset representative and, as discussed above, it leads to an x-dependent E_{11} transformation.

As a result we can combine the effect of the l_1 transformations and the x dependence E_{11} transformation together by taking an l_1 transformation with the parameter

$$\begin{aligned}
b_M(x) &= b_M + x^a a_{a,M} \\
b_{a,\alpha}(x) &= b_{a,\alpha} + x^b a_{ba,\alpha} \\
b_{ab,M\alpha}(x) &= b_{ab,M\alpha} + x^c a_{cab,M\alpha} \\
b_{abc,[\delta\epsilon]} &= b_{abc,[\delta\epsilon]} + x^d a_{dabc,[\delta\epsilon]} \quad .
\end{aligned} \tag{C.15}$$

As noted above we have in addition the usual E_{11} transformations with the a parameters which are related to the above parameters by

$$\begin{aligned}
a_{a,M} &= \partial_a b_M(x) & a_{ab,\alpha} &= \frac{1}{2}\partial_a b_{b,\alpha}(x) \\
a_{abc,M\alpha} &= \frac{1}{3}\partial_a b_{bc,M\alpha}(x) & a_{abcd,[\delta\epsilon]} &= \frac{1}{4}\partial_a b_{bcd,[\delta\epsilon]}(x) \quad .
\end{aligned} \tag{C.16}$$

Thus all the transformations can be expressed in terms of the x-dependent parameters $b(x)$.

The resulting transformations of the fields are given by

$$\begin{aligned}
\delta A_{a,M} &= \partial_a b_M(x) - g b_M(x) A_{a,N} X^{MN}{}_P - g b_{a,\alpha}(x) W^\alpha_M \qquad \text{(C.17)}\\
\delta A_{a_1a_2,\gamma} &= \frac{1}{2}\partial_{[a_1} b_{a_2],\gamma}(x) + \frac{1}{2}\partial_{[a_1|} b_M(x) A_{|a_2],N} (D_\gamma)^{MN} - g b_M(x) \Theta^M_\alpha f^{\alpha\beta}{}_\gamma A_{a_1a_2,\beta}\\
&\quad -\frac{1}{2} g b_{a_1,\alpha}(x) W^\alpha_M A_{a_2,N} D^{MN}_\gamma - \frac{1}{3} g b_{a_1a_2,M\alpha}(x) (\Theta^M_\beta f^{\beta\alpha}{}_\gamma - 2W^\alpha_N D^{MN}_\gamma)\\
\delta A_{a_1a_2a_3,M\alpha} &= \frac{1}{3}\partial_{[a_1} b_{a_2a_3],M\alpha}(x) + \partial_{[a_1|} b_M(x) A_{|a_2a_3],\alpha} - \frac{1}{6} A_{[a_1|,M} A_{|a_2|,N} \partial_{|a_3]} b_P(x) (D_\alpha)^{NP}\\
&\quad -g b_N(x) \Theta^N_\beta (D^\beta)_M{}^P A_{a_1\ldots a_3,P\alpha} - g b_N(x) \Theta^N_\beta f^{\beta\gamma}{}_\alpha A_{a_1\ldots a_3,P\gamma}\\
&\quad -g W^\beta_M b_{a_1,\beta}(x) A_{a_2a_3,\alpha} - \frac{1}{6} g W^\beta_N b_{a_1,\beta}(x) A_{a_2,P} A_{a_3,M} D^{NP}_\alpha\\
&\quad -g b_{a_1\ldots a_3,[\beta\alpha]}(x) W^\beta_M\\
\delta A_{a_1\ldots a_4,[\delta\epsilon]} &= \frac{1}{4} c^{M,N\beta}_{[\delta\epsilon]} \partial_{a_1} b_M(x) A_{a_2\ldots a_4,N\beta} - \frac{1}{24} D^{NP}_\alpha c^{M,Q\alpha}_{[\delta\epsilon]} A_{a_1,M} A_{a_2,Q} A_{a_3,N} \partial_{a_4} b_P(x)\\
&\quad +\frac{1}{4} D^{NP}_\delta A_{a_1,N} \partial_{a_2} b_P(x) A_{a_3a_4,\epsilon} + \frac{1}{4}\partial_{a_1} b_{a_2,\delta}(x) A_{a_3a_4,\epsilon} + \frac{1}{4}\partial_{a_1} b_{a_2\ldots a_4,[\delta\epsilon]}(x).
\end{aligned}$$

The field strengths are just the objects which are covariant under the above $E_{11} \otimes l_1$ transformations which at order g^0 agree with those of the massless theory. Thus the order g^1 variation of $F^{(0)}$, i.e. the variation of the massless field strengths, must cancel the order g^0 variation of the order g^0 variation of the order g^1 part of the field strength, $F^{(1)}$. This computation implies that the field strengths of the gauged theory are given by

$$\begin{aligned}
F_{a_1a_2,M} &= 2\partial_{[a_1} A_{a_2],M} + g X^{NP}{}_M A_{[a_1N} A_{a_2]P} + 4g A_{a_1a_2\alpha} W^\alpha_M\\
F_{a_1\ldots a_3,\alpha} &= 3\partial_{[a_1} A_{a_2a_3],\alpha} + \frac{3}{2}(\partial_{[a_1} A_{a_2,M}) A_{a_3],N} (D_\alpha)^{MN}\\
&\quad +\frac{1}{2} g A_{[a_1,M} A_{a_2,N} A_{a_3],P} X^{[MN}_Q D^{P]Q}_\alpha + 6g A_{[a_1a_2,\beta} A_{a_3],M} W^\beta_N D^{MN}_\alpha\\
&\quad +3g(\Theta^M_\gamma f^{\gamma\beta}{}_\alpha + 2g W^\beta_N D^{MN}_\alpha) A_{a_1\ldots a_3,M\beta}\\
F_{a_1\ldots a_4,T\eta} &= 4\partial_{[a_1} A_{a_2\ldots a_4],T\eta} - 4(\partial_{[a_1} A_{a_2a_3,\eta}) A_{a_4],T} - \frac{2}{3}(\partial_{[a_1} A_{a_2,N}) A_{a_3,P} A_{a_4],T} (D_\eta)^{NP}\\
&\quad -16 g W^\alpha_T A_{a_1\ldots a_4,[\alpha\eta]} + 4g(\Theta^M_\beta f^{\beta\alpha}{}_\eta + 2g W^\alpha_P D^{MP}_\eta) A_{[a_1\ldots a_3,M\alpha} A_{a_4]T}\\
&\quad -4g W^\alpha_M A_{[a_1a_2,\alpha} A_{a_3a_4],\eta} + 4g W^\alpha_P D^{PM}_\eta A_{[a_1a_2,\alpha} A_{a_3,M} A_{a_4],T}\\
&\quad +\frac{1}{6} g X^{MN}{}_R D^{RP}_\eta A_{[a_1,M} A_{a_2,N} A_{a_3,P} A_{a_4],T} \quad . \qquad \text{(C.18)}
\end{aligned}$$

The field strengths and the gauge transformations agree with those found in section 4, as one can see comparing eqs. (4.31) and (4.32) with eqs. (C.18) and (C.17), if we identify the gauge parameters of section 4 as

$$\Lambda_M = b_M(x) \quad \Lambda_{a,\alpha} = \frac{1}{2} b_{a,\alpha} \quad \Lambda_{ab,M\alpha} = \frac{1}{3} b_{ab,M\alpha} \quad \Lambda_{abc,[\alpha\beta]} = \frac{1}{4} b_{abc,[\alpha\beta]} \quad . \qquad \text{(C.19)}$$

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